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THE
PRINCIPLES OF FLUXIONS:

DESIGNED

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THE

PRINCIPLES OF FLUXIONS.

SECTION I.

DEFINITIONS.

ARTICLE 1. Every quantity is here conſidered as generated by motion; a line by the motion of a point; a ſurface by the motion of a line; a ſolid by the motion of a ſurface.*

2. The quantity thus generated is called the *fluent*, or *flowing* quantity.

3. The *velocities* with which flowing quantities increaſe or decreaſe at any point of time are called the *fluxions* of thoſe quantities at that inſtant.

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* SIR I. NEWTON, in the Introduction to his *Quadrature of Curves*, obſerves that "theſe geneſes really take place in the nature of things, and are daily ſeen in the motion of bodies. And after this manner, the ancients, by drawing moveable right lines along immoveable right lines, taught the geneſis of rectangles."

Cor. 1. As the velocities are in proportion to the increments or decrements *uniformly* generated in a given time, ſuch increments or decrements will repreſent the fluxions.*

Cor. 2. Hence, as any given time may be aſſumed, the fluxion is not an *abſolute* but a *relative* quantity. When we have ſeveral cotemporary fluxions, we may aſſume one fluxion what we pleaſe, and thence determine the values of the others. Thus, if x and y increaſe uniformly, and if x increaſe by p in the time that y increaſes by q, then the cotemporary increments of x and y will be p and q, $2p$ and $2q$, $3p$ and $3q$, &c. hence if p be aſſumed the fluxion of x, the fluxion of y will be q; if the former fluxion be $2p$, the latter will be $2q$, &c. &c.

Cor. 3. A *conſtant* quantity has no fluxion.

4. The firſt letters, a, b, c, &c. of the alphabet are uſually put for conſtant quantities, and the laſt, v, w, x, y, z, for variable ones; and they are to be thus underſtood, unleſs the contrary be expreſſed.

5. The fluxion of a ſimple quantity, as x, is expreſſed by placing a point over it, thus $\dot{x}$.

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* This is agreeable to Sir I. Newton's ideas on the ſubject. He ſays, "I ſought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling theſe velocities of the motions or increments, *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions."

Introd. to *Quad. Curves.*

TO FIND THE FLUXIONS OF QUANTITIES.

Prop. I.

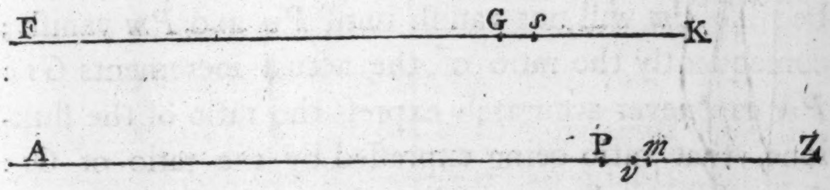
If two quantities increaſe or decreaſe uniformly, the increments or decrements generated in a given time will be as their fluxions.

6. This appears from Art. 3. Cor. 1.

Prop. II.

If one quantity increaſe uniformly, and another of the ſame kind increaſe with an accelerated or retarded velocity, and two increments be aſſumed which are generated in the ſame time; if thoſe increments be diminiſhed till they vaniſh, that ratio to which they approach as their limit is the ratio of the fluxions of thoſe quantities.

7. Let the line FK be deſcribed with an uniform velocity, and AZ with an accelerated velocity, and let the increments Gs, Pm be generated in the ſame time; let alſo Pv be the increment that would have been generated in the ſame time if the velocity at P had been continued uniform; then, by Prop. 1. the fluxions of FK, AZ at the points G and P will be repreſented by Gs and Pv. Let v be the velocity with which Gs is deſcribed, and V the velocity with



which Pv is deſcribed, and let the velocity at m be $V+r$; then vm is the increment which is deſcribed in conſequence of the increaſe r of velocity ſince the deſcribing point left P. Now let Pm be deſcribed by the *uniform* velocity $V+w$ in the ſame time that Pv and Pm are deſcribed; then it is manifeſt that this uniform velocity muſt be between the velocities at P and at m, that is, $V+w$ is greater than V and leſs than $V+r$, or w is greater than o and leſs than r. Alſo, ſince the ſpaces deſcribed in the ſame time are as the velocities, $V : V+w :: Pv : Pm$. Now diminiſh the times in which theſe increments are deſcribed; then as the points v and m approach to P, Pv will continue to be deſcribed with the uniform velocity V; but r will be diminiſhed, and by diminiſhing the time till it becomes indefinitely ſmall, r will become indefinitely ſmall; but vm is deſcribed in conſequence of this increaſe r of velocity; hence when r becomes indefinitely ſmall in reſpect to V, the ſpace vm muſt become indefinitely ſmall in reſpect to Pv; therefore the ratio of $Pv : Pm$ is, in that ſtate, indefinitely near to a ratio of equality; but it is manifeſt that it never can become accurately a ratio of equality, becauſe vm will not vaniſh until Pv and Pm vaniſh; conſequently the ratio of the actual increments $Gs : Pm$ can never accurately expreſs the ratio of the fluxions, that ratio being expreſſed by the ratio of $Gs : Pv$. Let us then conſider what ratio $Pv : Pm$ approaches to as it's *limit*, when we make the time in which the increments are deſcribed, and conſequently the increments themſelves, vaniſh. In every ſtate of theſe increments, $V : V+w :: Pv : Pm$; and by continually diminiſhing the time, and conſequently the increments, we diminiſh r, and conſequently w, but V remains conſtant; it is manifeſt therefore that the ratio of $V : V+w$, and conſequently that of $Pv : Pm$, continually approaches to a ratio of equality, agreeable to what we have already ſhown; and when the time, and conſequently the increments, become actually $=0$, then $r=0$; conſequently $w=0$; hence the *limit* of the ratio of $Pv : Pm$ becomes that of $V : V$, a ratio of equality.* Hence the *limit* of the ratio of $Gs : Pm$ is the ſame as the *limit* of the ratio of $Gs : Pv$, or as $Gs : Pv$, that ratio being conſtant; that is, *the limiting ratio of the increments is the ratio of the fluxions.*

The ſame is manifeſtly true for the limiting ratio of the decrements of two quantities; for conceiving the deſcribing points to move backwards, and to be retarded by the ſame law, the decrements sG, mP in this caſe become the ſame as the increments in the other; conſequently their *limiting* ratio will expreſs the ratio of the fluxions at G and P, or the rate at which FG, AP are, at that inſtant, decreaſing.

Cor. 1. Hence the *limiting* ratio of the increments or decrements of two quantities which are *both* generated by variable velocities will be the ratio of their fluxions. And as the velocities with which theſe two lines increaſe or decreaſe may be made to agree with the rate of increaſe or decreaſe of any two like quantities, the propoſition muſt be true for quantities of any kind.

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* By keeping the ratio of the vaniſhing quantities thus expreſſed by finite quantities, it removes the obſcurity which may ariſe when we conſider the quantities themſelves; this is agreeable to the reaſoning of SIR I. NEWTON in his *Principia, Lib. I. Sect. I. Lem.* 7. 8. 9.

Cor. 2. As the *limiting* ratio of the increments is the ratio of the fluxions, it is manifeſt that when the increments are in an increaſing or decreaſing ſtate, the fluxions will be increaſing or decreaſing.

8. It has been ſaid, that when the increments are actually vaniſhed, it is abſurd to talk of any ratio between them. It is true; but we ſpeak not here of any ratio then exiſting between the quantities, but of that ratio to which they have approached as their *limit*; and that ratio ſtill remains. Thus, let the increments of two quantities be denoted by ax^2+mx and bx^2+nx; then the *limit* of their ratio, when $x=0$, is $m:n$; for $ax^2+mx : bx^2+nx :: ax+m : bx+n ::$ (when $x=0$) $m:n$. As the quantities therefore approach to nothing, the ratio approaches to that of $m:n$ as the *limit*. Hence if $m=n$, the *limit* of this ratio is a ratio of equality. We muſt therefore be careful to diſtinguiſh between the ratio of two evaneſcent quantities, and the *limit* of their ratio; the former ratio never arriving at the latter, as the quantities vaniſh at the inſtant that ſuch a circumſtance is about to take place.

PROP. III.

If the fluxion of x be denoted by $\dot{x}$, the fluxion of ax will be $a\dot{x}$.

9. For if x increaſe uniformly, ax will alſo increaſe uniformly and a times as faſt; hence, by Prop. 1. the fluxion of the latter will be a times greater than that of the former, or it will be $a\dot{x}$.

Cor. Hence in taking the fluxion of a variable quantity multiplied into a conſtant one, the conſtant multiplier is retained.

Prop. IV.

The fluxion of $x \pm a$ *is* $\dot{x}$.

10. For a being conſtant, and only connected to x by the ſigns + or −, it does not affect the *increaſe* or *decreaſe* of the quantity; therefore the fluxion is the ſame as the fluxion of x, or it is $\dot{x}$.

Cor. Hence conſtant quantities connected to variable ones by the ſigns + or −, diſappear when the fluxions are taken.

Prop. V.

Given ($\dot{x}$) *the fluxion of* x, *to find the fluxion of* x^n, n *being a whole number.*

11. Let x increaſe uniformly by v and become $x+v$, then will x^n become $\overline{x+v}|^n$; but by the Elements of Algebra, Art. 232. $\overline{x+v}|^n = x^n + nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ and if from this quantity we take x^n, there remains $nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ for the cotemporary increment of x^n; but although x increaſe uniformly by v, x^n does not increaſe uniformly; for if in the increment of x^n we ſubſtitute 1, 2, 3, &c. for v, and take the differences of the reſults, theſe differences will not be equal; hence to get the ratio of the fluxion of x to the fluxion of x^n we muſt, according to Prop. 2. take the *limiting* ratio of the increments. Now the increment of x : the increment of x^n :: $v : nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c. :: 1 : nx^{n-1} + n.\frac{n-1}{2}x^{n-2}v + \&c.$ and to get the *limiting* ratio of theſe increments we

muſt make $v=0$, in which caſe the ratio becomes $1 : nx^{n-1}$, which therefore expreſſes the ratio of the fluxion of x to the fluxion of x^n; but $\dot{x}$ denotes the fluxion of x, therefore $nx^{n-1}\dot{x}$ repreſents the cotemporary fluxion of x^n.

If $n=0$, $x^n=1$ a conſtant quantity; therefore by Art. 3. Cor. 3. it has no fluxion.

Prop. VI.

To find the fluxion of $x^{\frac{n}{m}}$, *m and n being any whole numbers.*

12. Put $y=x^{\frac{n}{m}}$, then $y^m = x^n$; hence by taking the fluxions, $my^{m-1}\dot{y}=nx^{n-1}\dot{x}$, $\therefore\ \dot{y}=\frac{nx^{n-1}\dot{x}}{my^{m-1}}$ = (by ſubſtituting for y it's value in terms of x) $\frac{nx^{n-1}\dot{x}}{mx^{\frac{nm-n}{m}}} =$ $\frac{nx^{n-1}\dot{x}}{mx^{n-\frac{n}{m}}} = \frac{n}{m}\times x^{\frac{n}{m}-1}\dot{x}$.

Cor. Let the root be a compound quantity as a^m+x^m, to find the fluxion of $\overline{a^m+x^m}\big|^{\frac{1}{n}}$. Put $y=\overline{a^m+x^m}\big|^{\frac{1}{n}}$, then $y^n=a^m+x^m$, and $ny^{n-1}\dot{y}=mx^{m-1}\dot{x}$, hence $\dot{y}=\frac{mx^{m-1}\dot{x}}{ny^{n-1}}=\frac{mx^{m-1}\dot{x}}{n\times\overline{a^m+x^m}\big|^{\frac{n-1}{n}}}=\frac{1}{n}\times\overline{a^m+x^m}\big|^{\frac{1-n}{n}}\times$ $mx^{m-1}\dot{x}=\frac{1}{n}\times\overline{a^m+x^m}\big|^{\frac{1}{n}-1}\times mx^{m-1}\dot{x}$.

13. Hence it appears, that whether the root be a ſimple or a compound quantity, the fluxion of any power thereof is found by the following

RULE:

RULE:

Multiply by the index, diminiſh the index by unity, and multiply by the fluxion of the root.

Examples.

Ex. 1. The fluxion of x^9 is $9x^8\dot{x}$.

Ex. 2. The fluxion of $3y^5$ is $15y^4\dot{y}$.

Ex. 3. The fluxion of $\frac{3}{2}y^{\frac{4}{7}}$ is $\frac{12}{14}y^{-\frac{3}{7}}\dot{y}=\frac{6\dot{y}}{7y^{\frac{3}{7}}}$.

Ex. 4. The fluxion of $\frac{5}{9}x^{\frac{7}{11}}$ is $\frac{35}{99}x^{-\frac{4}{11}}\dot{x}=\frac{35\dot{x}}{99x^{\frac{4}{11}}}$.

Ex. 5. The fluxion of $\frac{4}{7}x^{\frac{11}{9}}$ is $\frac{44}{63}x^{\frac{2}{9}}\dot{x}$.

Ex. 6. What is the fluxion of $\overline{a^2+x^2}|^3$?

Here the root is a^2+x^2, and its fluxion $2x\dot{x}$; hence the fluxion required is $3\times\overline{a^2+x^2}|^2\times 2x\dot{x} = \overline{a^2+x^2}|^2\times 6x\dot{x}$.

Ex. 7. What is the fluxion of $\sqrt{a^2+x^2}$, or of $\overline{a^2+x^2}|^{\frac{1}{2}}$?

Here the root is a^2+x^2, and it's fluxion $2x\dot{x}$; hence the fluxion is $\frac{1}{2}\times\overline{a^2+x^2}|^{-\frac{1}{2}}\times 2x\dot{x}=\frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$.

Ex. 8. What is the fluxion of $\overline{x^2+y^2}|^{\frac{3}{2}}$?

Here the root is x^2+y^2, and it's fluxion $2x\dot{x}+2y\dot{y}$; hence the fluxion required is $\frac{3}{2}\times\overline{x^2+y^2}|^{\frac{1}{2}}\times\overline{2x\dot{x}+2y\dot{y}}$ $=3\times\overline{x^2+y^2}|^{\frac{1}{2}}\times\overline{x\dot{x}+y\dot{y}}$.

Ex. 9. What is the fluxion of $\overline{x+y}|^2$?

Here the root is $x+y$, and it's fluxion $\dot{x}+\dot{y}$; hence the fluxion required is $2\times\overline{x+y}\times\overline{\dot{x}+\dot{y}}$.

Ex. 10. What is the fluxion of $\overline{a^5+x^5}|^{\frac{1}{2}}$?

Here the root is a^5+x^5, and it's fluxion $5x^4\dot{x}$;

hence the fluxion required is $\frac{1}{2}\times\overline{a^5+x^5}|^{-\frac{1}{2}}\times 5x^4\dot{x}=$ $\frac{5x^4\dot{x}}{2\times\overline{a^5+x^5}|^{\frac{1}{2}}}$.

Ex. 11. What is the fluxion of $\frac{1}{\overline{a^2+x^2}|^{\frac{5}{9}}}$?

This quantity becomes $\overline{a^2+x^2}|^{-\frac{5}{9}}$, and the root is a^2+x^2, whoſe fluxion is $2x\dot{x}$; hence the fluxion required is $-\frac{5}{9}\times\overline{a^2+x^2}|^{-\frac{14}{9}}\times 2x\dot{x}=\frac{-10x\dot{x}}{9\times\overline{a^2+x^2}|^{\frac{14}{9}}}$. In like manner, bring any quantity from the denominator up to the numerator, by changing the ſign of the index, and then proceed by the rule.

Ex. 12. What is the fluxion of $\overline{ax^2+by^3+cz^4}|^{\frac{7}{3}}$?

Here the root is $ax^2+by^3+cz^4$, and it's fluxion $2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}$; hence the fluxion required is $\frac{7}{3}\times\overline{ax^2+by^3+cz^4}|^{\frac{4}{3}}\times\overline{2ax\dot{x}+3by^2\dot{y}+4cz^3z}$.

13. What is the fluxion of $\sqrt{x^2+\sqrt{a^2+y^2}}$?

Put $z=\sqrt{x^2+\sqrt{a^2+y^2}}$, then $z^2=x^2+\sqrt{a^2+y^2}$; now the fluxion of $\sqrt{a^2+y^2}$, or of $\overline{a^2+y^2}|^{\frac{1}{2}}$, is $\frac{1}{2}\times\overline{a^2+y^2}|^{-\frac{1}{2}}\times 2y\dot{y}=\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}$; hence $2z\dot{z}=2x\dot{x}+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}$, therefore $\dot{z}=\frac{2x\dot{x}+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}}{2z}=\frac{2x\dot{x}+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}}{2\sqrt{x^2+\sqrt{a^2+y^2}}}$.

PROP. VII.

To find the fluxion of a product xy.

14. The fluxion of $\overline{x+y}^{2}$, by the laſt rule, is $2\times\overline{x+y}\times\overline{\dot{x}+\dot{y}}=2\dot{x}\dot{x}+2x\dot{y}+2y\dot{x}+2y\dot{y}$; alſo $\overline{x+y}^{2}$ $=x^{2}+2xy+y^{2}$, whoſe fluxion is $2x\dot{x}+$ the fluxion of $2xy+2y\dot{y}$; make theſe two values of the fluxion of $\overline{x+y}^{2}$ equal to each other; omit the firſt and laſt terms which are common to both, and we have the fluxion of $2xy=2x\dot{y}+2y\dot{x}$; hence the fluxion of xy is $x\dot{y}+y\dot{x}$.

Otherwiſe thus. If we ſuppoſe x conſtant, the fluxion of xy is $x\dot{y}$ by Prop. 3; and if we ſuppoſe y conſtant, the fluxion is $y\dot{x}$; hence if neither be conſtant, the fluxion is $x\dot{y}+y\dot{x}$.

Cor. Hence we may find the fluxion of xyz. For if $v=xyz$, and $w=xy$, then $v=wz$, $\therefore\dot{v}=w\dot{z}+z\dot{w}$; but $w=xy$, $\therefore\dot{w}=x\dot{y}+y\dot{x}$; ſubſtitute theſe values for w and $\dot{w}$, and we get $\dot{v}=xy\dot{z}+zxy+zy\dot{x}$.

15. In like manner we proceed for any number of factors; hence the fluxion of the product of any number of quantities is found by the following

RULE:

Multiply the fluxion of each quantity into the product of all the reſt, and the ſum of all the products is the fluxion required.

Examples.

Ex. 1. The fluxion of $x^{2}y^{3}$ is $x^{2}\times 3y^{2}\dot{y}+y^{3}\times 2x\dot{x}$ $=3x^{2}y^{2}\dot{y}+2y^{3}x\dot{x}$.

Ex. 2. The fluxion of $y^{\frac{7}{2}}x^{\frac{5}{3}}z$ is $x^{\frac{5}{3}}z\times\frac{7}{2}y^{\frac{5}{2}}\dot{y}+y^{\frac{7}{2}}z\times$ $\frac{5}{3}x^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}=\frac{7}{2}x^{\frac{5}{3}}zy^{\frac{5}{2}}\dot{y}+\frac{5}{3}y^{\frac{7}{2}}zx^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}$.

Ex. 3. The fluxion of $w^{m}x^{n}y^{r}z^{s}$ is $mx^{n}y^{r}z^{s}w^{m-1}\dot{w}+nw^{m}y^{r}z^{s}x^{n-1}\dot{x}+rw^{m}x^{n}z^{s}y^{r-1}\dot{y}+sw^{m}x^{n}y^{r}z^{s-1}\dot{z}$.

Ex. 4. To find the fluxion of $x^2\times\overline{a^4+y^4}\big|^{\frac{3}{2}}$.

By the laſt rule, the fluxion of $\overline{a^4+y^4}\big|^{\frac{3}{2}}$ is $\frac{3}{2}\times\overline{a^4+y^4}\big|^{\frac{1}{2}}\times 4y^3\dot{y}=6\times\overline{a^4+y^4}\big|^{\frac{1}{2}}\times y^3\dot{y}$; hence the fluxion required is $x^2\times 6\times\overline{a^4+y^4}\big|^{\frac{1}{2}}\times y^3\dot{y}+\overline{a^4+y^4}\big|^{\frac{3}{2}}\times 2x\dot{x}$.

Ex. 5. To find the fluxion of $\sqrt{a^2+x^2}\times\sqrt{b^2+y^2}$.

Find the fluxion of each part by the laſt rule, and the fluxion required is $\sqrt{a^2+x^2}\times\frac{y\dot{y}}{\sqrt{b^2+y^2}}+\sqrt{b^2+y^2}\times\frac{x\dot{x}}{\sqrt{a^2+x^2}}$.

16. It appears from this Prop. that the fluxion of xy conſiſts of two parts, $x\dot{y}$ and $y\dot{x}$, the former part ariſing from the increaſe of y by $\dot{y}$, and the latter from the increaſe of x by $\dot{x}$; but if x ſhould decreaſe whilſt y increaſes, then the fluxion, expreſſing the *increaſe* of xy upon the whole, will be $x\dot{y}-y\dot{x}$, being the increaſe *minus* the decreaſe. Hence to expreſs the rate at which any quantity *increaſes*, the fluxion of the parts which increaſe muſt be written with the ſign +, and thoſe which decreaſe with the ſign −. Now the increaſing quantity is conſidered as poſitive; but if a negative quantity increaſe in magnitude, it muſt be conſidered as a decreaſing quantity, and it's fluxion will be negative. In like manner a negative quantity decreaſing in magnitude muſt be conſidered as an increaſing quantity, and it's fluxion will be poſitive. If therefore the fluxions of increaſing quantities be written with the ſign +, and of decreaſing with −, whenever the fluxion of any quantity is poſitive it ſhows that quantity to be in an increaſing ſtate; and when negative, to be in a decreaſing ſtate.

PROP.

Prop. VIII.

To find the fluxion of a fraction $\frac{x}{y}$.

17. The quantity $\frac{x}{y} = xy^{-1}$, therefore the fluxion, by the laſt rule, is $y^{-1}\dot{x} - 1 \times xy^{-2}\dot{y} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{y\dot{x} - x\dot{y}}{y^2}$. Hence we find the fluxion of a fraction by the following

RULE:

From the fluxion of the numerator multiplied into the denominator, ſubtract the fluxion of the denominator multiplied into the numerator, and divide by the ſquare of the denominator.

Examples.

Ex. 1. The fluxion of $\frac{x^2}{y^3}$ is $\frac{2y^3x\dot{x} - 3x^2y^2\dot{y}}{y^6} = \frac{2yx\dot{x} - 3x^2\dot{y}}{y^4}$.

Ex. 2. The flux. of $\frac{x+y}{z^3}$ is $\frac{z^3 \times \overline{\dot{x}+\dot{y}} - \overline{x+y} \times 3z^2\dot{z}}{z^6}$ $= \frac{z \times \overline{\dot{x}+\dot{y}} - \overline{x+y} \times 3\dot{z}}{z^4}$.

Ex. 3. The flux. of $\frac{xy}{z^2}$ is $\frac{z^2 \times \overline{x\dot{y}+y\dot{x}} - xy \times 2z\dot{z}}{z^4}$ $= \frac{z \times \overline{x\dot{y}+y\dot{x}} - 2xy\dot{z}}{z^3}$.

Ex. 4. The fluxion of $\frac{1}{x^n}$ is $\frac{-nx^{n-1}\dot{x}}{x^{2n}} = -\frac{\dot{x}}{x^{n+1}}$.

Ex. 5. The fluxion of $\frac{\sqrt{a^2+x^2}}{\sqrt{b^2+y^2}}$ is

$$\frac{\overline{a^2+x^2}|^{-\frac{1}{2}}\times x\dot{x}\times\sqrt{b^2+y^2}-\overline{b^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}\times\sqrt{a^2+x^2}}{b^2+y^2}$$

$$=\frac{x\dot{x}}{\sqrt{a^2+x^2}\times\sqrt{b^2+y^2}}-\frac{\sqrt{a^2+x^2}\times y\dot{y}}{\overline{b^2+y^2}|^{\frac{3}{2}}}.$$

The putting of a quantity into fluxions is called the *direct* method of fluxions.

SCHOLIUM.

18. In queſtions of a geometrical and philoſophical nature, where we want to get the relation of the fluents from the fluxions, and in others where we want to find whether quantities are poſitive or negative from the relation of them to their fluxions, it is neceſſary to pay regard to the ſigns of the fluxions, as explained in Art. 16. But in putting equations into fluxions, as in the Problems de Maximis et Minimis, although one variable quantity may increaſe at the ſame time that another decreaſes, yet we may write the fluxion of each poſitive; for by writing it ſo in each equation in order to obtain the ſame fluxion from the different equations, the reſult will not be altered. In theſe, and ſuch like caſes, we may therefore make the fluxion of each quantity poſitive. We may further obſerve, that when any fluxion becomes negative according to the above rule, the quantity which exexpreſſes it's value becomes negative. For inſtance, if $r=$ the radius of a circle, $x=$ the verſed ſine, $y=$ the right ſine of an arc, then $y^2=2rx-x^2$, and $\dot{y}=\frac{r\dot{x}-x\dot{x}}{y}$; now for the firſt quadrant x and y increaſe,

and each fluxion is poſitive, and the value of $\dot{y}$ is poſitive, x being leſs than r; but in the ſecond quadrant y decreaſes and it's fluxion becomes negative, and it's value becomes negative, x being greater than r. This circumſtance is ſimilar to the caſe of a quantity paſſing through o and changing it's ſign, for $\dot{y}=o$ at the end of the quadrant.

19. When we compare the fluxions of two quantities by comparing the increments that would be *uniformly* generated in a given time, the quantities have been ſuppoſed to be homogeneous, there being no relation between thoſe which are not homogeneous; yet if, of two heterogeneous quantities, the *numerical value* of one be expreſſed in terms of the other, it is manifeſt that there will be no impropriety in expreſſing the fluxion of one in terms of the fluxion of the other. If one ſide of a right angled parallelogram be repreſented by 6 and the other by 9, we ſay $6\times9=54$ the area; our numerical operation is perfectly correct, but no one ever imagined that the units repreſented by 54 are homogeneous to the units repreſented by 6 and 9; if 6 and 9 repreſent inches in *length*, 54 will repreſent ſo many *ſquare* inches, or ſo many *ſquare* areas, the ſide of each of which is 1 inch in length. Or if a and x repreſent the two ſides, the area of the parallelogram will actually be ax, referring that quantity to it's proper units; although therefore there is no relation between the area and either of it's ſides, yet it is expreſſed in terms of the ſides. And if a be conſtant and x variable, the fluxion of the area will be $a\dot{x}$ by Prop. 3; if therefore $(\dot{x})$ the fluxion of the abſciſſa x

be 1 inch in *length*, the correſponding fluxion of the area will be *a ſquare* inches; if $\dot{x}$ be 2 inches in *length*, the fluxion of the area will be 2*a ſquare* inches. And in general, when we conſider any two quantities which are not homogeneous, although their fluxions, which are expreſſed by their increments *uniformly* generated in a given time, can have no relation to each other, if we carry our ideas no further than the increments themſelves; yet when we conſider the numerical values of theſe fluxions, the analytical expreſſion for one may be compriſed in terms of the other without any impropriety, and our concluſions will be perfectly juſt and correct, in the ſenſe in which the units of the reſpective quantities are underſtood, notwithſtanding the fluxions themſelves may be heterogeneous. SIR I. NEWTON, in his *Quadrature of Curves*, in finding the area of a curve, deſcribes a parallelogram on the abſciſſa (x), the other ſide (a) of which is conſtant; and then he compares the fluxion of the area of this parallelogram with the fluxion of the area of the curve, they being homogeneous quantities; and the fluxion of the area of the parallelogram being $a\dot{x}$, he gets the fluxion of the area of the curve. From what has been ſaid above, when we reduce theſe matters to calculation there appears to be no abſolute neceſſity for this; but it is more ſcientific to make the compariſon between homogeneous quantities, than between thoſe which are not homogeneous, and therefore the former method is always to be preferred in caſes where it can be applied, notwithſtanding the concluſions which are otherwiſe deduced are perfectly true and ſatisfactory.

20. The ingenious and juſtly celebrated Author of the *Analyſt* has endeavoured to ſhow, that the principles of fluxions, as delivered by it's Author, are not founded upon reaſoning ſtrictly logical and concluſive. He lays this down as a Lemma: "If you make any ſuppoſition, and in virtue thereof deduce any conſequence; if you deſtroy that ſuppoſition, every conſequence before deduced muſt be deſtroyed and rejected, ſo as from thence forward to be no more ſupplied or applied in the demonſtration." This, he thinks, is ſo plain as to need no proof. It may perhaps be admitted to be true, when we want to deduce the *abſolute* value of a quantity which is to be obtained in virtue of a ſuppoſition; but it is not true when we want to obtain the *relative* values of quantities. He ſeems not to have attended to the connection which there muſt neceſſarily be between the *limit* of a ratio and the *terms* of the ratio. It is agreed, that by diminiſhing the increments you approach to the ratio of the velocities which they had at the points from whence they began to be generated, and that by making them become indefinitely ſmall, you arrive at a ratio indefinitely near to that of the velocities at thoſe points. Now the *limit* of a ratio muſt depend upon the *terms* of the ratio, and the terms are obtained upon the ſuppoſition of the exiſtence of an increment; the *limit* therefore is obtained upon the ſuppoſition of the exiſtence of an increment; but the *limit* is a certain determinate invariable ratio, totally independent of the *magnitude* of the terms of the ratio, and conſequently of the increment, as appears by Art. 8. when we therefore deduce the *limit* by making the increments vaniſh, the *effect*

of it's exiſtence muſt ſtill remain in the *limit*. If the *exiſtence* of the *limit* depended upon the *exiſtence* of the *terms* of the ratio, the ſuppoſition which makes the latter vaniſh would neceſſarily make the former alſo vaniſh; but as that is not the caſe, the *limit*, which is obtained by making the terms become equal to nothing, contains an effect, after the increments are actually vaniſhed, which depends upon their having exiſted. The lemma therefore of the Author, however true it may be in ſome circumſtances, cannot be applied againſt the reaſoning upon which the Principles of Fluxions are founded. The Author admits the concluſions to be true. He ſays, "I have no controverſy about your concluſions, but only about your logic; and it muſt be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them." The above obſervations ſhow, not only that our concluſions are true, but that they are deduced by ſteps which are perfectly ſatisfactory and ſtrictly logical.

* * *

SECTION II.

ON THE MAXIMA AND MINIMA OF QUANTITIES.

Prop. IX.

TO determine the value of a quantity when it becomes a maximum or minimum.

21. The fluxion of a quantity being it's uniform increaſe or decreaſe in a given time, when it becomes a maximum or minimum it's fluxion muſt be $=0$, the quantity having, at that point of time, no further increaſe or decreaſe.

If any quantity be a maximum or minimum, any power or root of that quantity muſt then evidently be a maximum or minimum.

Any conſtant multiple, or part of a quantity which is a maximum or minimum, muſt be a maximum or minimum; therefore when it's fluxion is made $=0$, the conſtant multiplier may be neglected.

It is further manifeſt, that for a quantity to become a maximum, it muſt firſt increaſe and then decreaſe; and to become a minimum, it muſt firſt decreaſe and then increaſe.

Examples.

Ex. 1. To divide a given number a into two parts, x, y, ſo that $x^m y^n$ may be a maximum.

Since $x+y=a$, and $x^m y^n=\text{max.}$ the fluxion of each $=0$, the former becauſe it is conſtant, and the latter becauſe it is a maximum; $\therefore\ \dot{x}+\dot{y}=0$, and $my^n x^{m-1}\dot{x}+nx^m y^{n-1}\dot{y}=0$; hence $\dot{x}=-\dot{y}$, and $\dot{x}=-\dfrac{nx^m y^{n-1}\dot{y}}{my^n x^{m-1}}=-\dfrac{nx\dot{y}}{my}$; therefore $-\dot{y}=-\dfrac{nx\dot{y}}{my}$, or $my=nx$, and $m:n::x:y$. Now $y=\dfrac{nx}{m}$; $\therefore\ x+\dfrac{nx}{m}=a$, consequently $x=\dfrac{ma}{m+n}$, and $y\left(=\dfrac{nx}{m}\right)=\dfrac{na}{m+n}$.

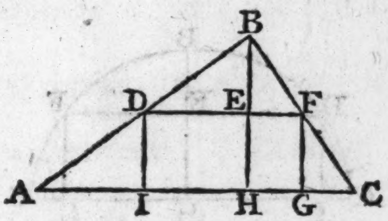
If $m=n$, the two parts are equal.

Hence to divide a quantity a into three parts, x, y, z, ſo that xyz may be a max. the parts muſt be equal. For ſuppoſe x to remain conſtant, and y, z, to vary; the product yz, and conſequently xyz, will be greateſt when $y=z$. Or if y remain conſtant, the product xz, and conſequently yxz, will be greateſt when $x=z$. Thus it appears that the parts muſt be equal. And in like manner it may be ſhown that whatever be the number of parts, they will be equal.

Ex. 2. To inſcribe the greateſt parallelogram $DFGI$ in the given triangle ABC.

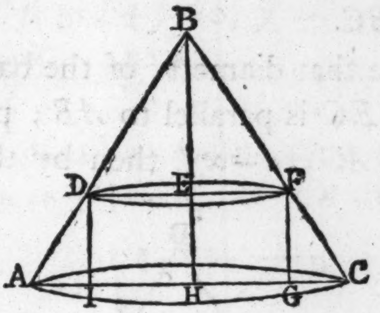
Draw $BH \perp AC$; put $AC=a$, $BH=b$, $BE=x$; then $EH=b-x$; and by ſim. Δs, $b:a::x:DF=\dfrac{ax}{b}$; hence the area $DFGI=\dfrac{ax}{b}\times\overline{b-x}=\text{max.}$ or $x\times\overline{b-x}$

$=bx-x^2=$ max. $\therefore b\dot{x}-2x\dot{x}=0$; hence $x=\frac{1}{2}b$, therefore $EH=\frac{1}{2}BH$,



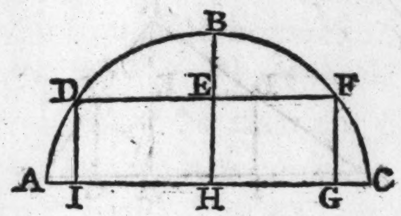
Ex. 3. Let *ABC* repreſent a cone, *AC* the diameter of the baſe; to inſcribe in it the greateſt cylinder *DFGI*.

Put $p=,78539$, &c. then, (the ſame notation remaining), it will appear when we come to treat on the method of finding the areas of curves, that $\frac{pa^2x^2}{b^2}=$ the area of the end *DEF* of the cylinder; hence the content of the cylinder $=\frac{pa^2x^2}{b^2}\times\overline{b-x}=$ max. or $x^2\times\overline{b-x}=bx^2-x^3=$ max. $\therefore 2bx\dot{x}-3x^2\dot{x}=0$, hence $x=\frac{2}{3}b$; therefore $EH=\frac{1}{3}BH$.



Ex. 4. To inſcribe the greateſt parallelogram *DFGI* in a given parabola *ABC*.

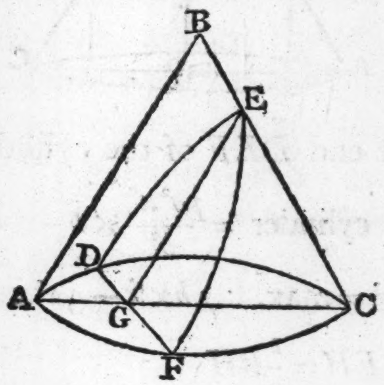
Put $BH=a$, $p=$ the parameter, $x=BE$, then by the property of the parabola, $DE^2=px$, $\therefore DE=p^{\frac{1}{2}}x^{\frac{1}{2}}$,



and $DF=2p^{\frac{1}{2}}x^{\frac{1}{2}}$; hence the area $DFGI=2p^{\frac{1}{2}}x^{\frac{1}{2}}\times\overline{a-x}$ $=$ max. or $x^{\frac{1}{2}}\times\overline{a-x}=ax^{\frac{1}{2}}-x^{\frac{3}{2}}=$ max. $\therefore \frac{1}{2}ax^{-\frac{1}{2}}\dot{x}-\frac{3}{2}x^{\frac{1}{2}}\dot{x}=0$, hence $\frac{a}{x^{\frac{1}{2}}}=3x^{\frac{1}{2}}$, or $a=3x$, $\therefore x=\frac{1}{3}a$, consequently $EH=\frac{2}{3}BH$.

Ex. 5. To cut the greateſt parabola DEF from the given cone ABC.

Let AGC be that diameter of the baſe which is $\perp$ to DGF; now EG is parallel to AB; put $AC=a$, $AB=b$, $CG=x$, $AG=a-x$; then by the property of



the circle, $DG=\sqrt{ax-x^2}$, $\therefore DF=2\sqrt{ax-x^2}$; alſo by ſim. Δs, $a : b :: x : GE = \frac{bx}{a}$; hence we have the area of the parabola $= \frac{2}{3} \times \frac{bx}{a} \times 2\sqrt{ax - x^2} =$ max. hence $x\sqrt{ax - x^2} =$ max. or $x^2 \times \overline{ax - x^2} = ax^3 - x^4 =$ max. $\therefore 3ax^2\dot{x} - 4x^3\dot{x} = 0$, $\therefore 3a = 4x$, and $x = \frac{3}{4}a$.

Ex. 6. Given $x + y + z = a$, and xy^2z^3 a maximum, to find x, y, z.

As x, y, z, muſt have ſome certain determinate values to anſwer theſe conditions, let us ſuppoſe ſuch a value of y to remain conſtant, whilſt x and z vary till they anſwer the conditions, and then $\dot{x} + \dot{z} = 0$ and $z^3\dot{x} + 3xz^2\dot{z} = 0$; hence $\dot{x} = -\dot{z} = -\frac{3xz^2\dot{z}}{z^3} = -\frac{3x\dot{z}}{z}$, $\therefore z = 3x$. Now let us ſuppoſe the value of z to remain conſtant, and x and y to vary, ſo as to ſatisfy the conditions; then $\dot{x} + \dot{y} = 0$, $y^2\dot{x} + 2xy\dot{y} = 0$; hence $\dot{x} = -\dot{y} = -\frac{2xy\dot{y}}{y^2} = -\frac{2x\dot{y}}{y}$, $\therefore y = 2x$; ſubſtitute in the given equation, theſe values of y and z in terms of x, and $x + 2x + 3x = a$, or $6x = a$, hence $x = \frac{1}{6}a$; $\therefore y = \frac{1}{3}a$, $z = \frac{1}{2}a$. In like manner, whatever be the number of unknown quantities, make any one quantity variable with each of the reſt, and the values of each in terms of that one quantity will be obtained; and by ſubſtituting the values of each in terms of that one in the given equation, you will get the value of that quantity, and from thence the values of the others.

Ex. 7. To find when y is a max. in $\overline{x^3 + y^3}|^2 = a^4x^2$.

Take the fluxion of both ſides, and $2 \times \overline{3x^2\dot{x}+3y^2\dot{y}} \times \overline{x^3+y^3} = 2a^4x\dot{x}$; but when y is a maximum, $\dot{y}=0$; hence $6x^2\dot{x} \times \overline{x^3+y^3} = 2a^4x\dot{x}$, $\therefore \overline{x^3+y^3} = \frac{a^4}{3x}$, and $\overline{x^3+y^3}^2 = \frac{a^8}{9x^2}$; hence $a^4x^2 = \frac{a^8}{9x^2}$, $\therefore x^4 = \frac{a^4}{9}$, and $x = \frac{a}{\sqrt{3}}$; hence $y^3 (= a^2x - x^3) = \frac{a^3}{\sqrt{3}} - \frac{a^3}{3^{\frac{3}{2}}} = a^3 \times \overline{\frac{1}{\sqrt{3}} - \frac{1}{3^{\frac{3}{2}}}} = a^3 \times \frac{2}{3\sqrt{3}}$; $\therefore y = a\sqrt[3]{\frac{2}{3\sqrt{3}}}$.

Otherwiſe. As $y^3 = a^2x - x^3$, $\therefore 3y^2\dot{y} = a^2\dot{x} - 3x^2\dot{x} = 0$, becauſe $\dot{y}=0$, $\therefore x = \frac{a}{\sqrt{3}}$.

Ex. 8. To determine at what angle the wind muſt ſtrike againſt the ſails of a mill, ſo that the effect to put it in motion may be the greateſt poſſible.

Put $x =$ the coſine of the angle, then $1 - x^2 =$ the ſquare of the ſine, radius being unity; hence (by the Principles of Hydroſtatics) the effect is as $x \times \overline{1-x^2} = x - x^3$, which is to be a maximum; $\therefore \dot{x} - 3x^2\dot{x} = 0$, hence $x = \sqrt{\frac{1}{3}}$ the coſine of $54°.\ 44'$.

Ex. 9. Given two elaſtic bodies A and C, to find an intermediate body x, ſo that the motion communicated from A to C through x may be the greateſt poſſible.

Put $a =$ the given velocity of A, $w =$ the velocity communicated to C, and z that to x; then (by Mechanics)

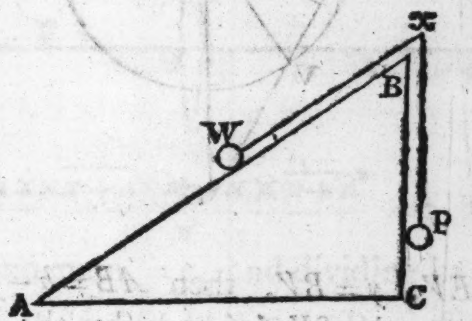
$$A + x : 2A :: a : w$$
$$x + C : 2x :: w : z$$

$\therefore$ comp. $Ax + x^2 + AC + Cx : 4Ax :: a : z$, or

$A+x+\frac{AC}{x}+C : 4A :: a : z$; now as the two middle terms are conſtant, the laſt term varies inverſely as the firſt; and as the laſt is to be a maximum, the firſt muſt be a minimum; therefore it's fluxion $\dot{x}-\frac{AC\dot{x}}{x^2}=0$; hence $x^2=AC$, and $A : x :: x : C$.

Ex. 10. Given the altitude BC of an inclined plane, to find it's length, ſo that a weight P acting upon another W in a line parallel to the plane, may draw it up through AB in the leaſt time.

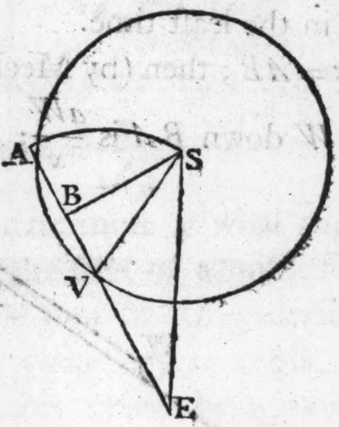
Put $a=BC$, $x=AB$; then (by Mechanics) the accelerating force of W down BA is $\frac{aW}{x}$; hence the moving force of the two bodies is $P-\frac{aW}{x}=\frac{Px-aW}{x}$; therefore the accelerating force $=\frac{Px-aW}{\overline{P+W}\times x}$; and the time of deſcribing AB varies as $\sqrt{\frac{AB}{\text{ac. for.}}}$ or as $\sqrt{\frac{\overline{P+W}\times x^2}{Px-aW}}=$ min. or $\frac{x^2}{Px-aW}=$ min. ∴ $\frac{2x\dot{x}\times\overline{Px-aW}-P\dot{x}\times x^2}{\overline{Px-aW}|^2}=0$; but when a fraction vaniſhes,



vaniſhes, it's numerator $=0$; hence $2Px^2\dot{x}-2aWx\dot{x}-Px^2\dot{x}=0$, or $Px^2=2aWx$, $\therefore x=\frac{2aW}{P}$.

Ex. 11. To find the poſition of the planet Venus, when it gives the greateſt quantity of light to the Earth.

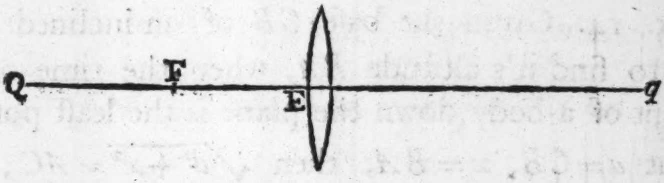
Let S be the Sun, E the Earth, V Venus, produce EV, on which let fall the $\perp$ SB, and with the center V deſcribe the circular arc SA. Put $a=SE$, $b=SV=$



AV, $x=EV$, $y=BV$, then $AB=b-y$ the verſed ſine of the angle SVA; and (by the Principles of Aſtronomy) the quantity of light received at the Earth from Venus varies as $\frac{b-y}{x^2}=\frac{b}{x^2}-\frac{y}{x^2}=$ max. Now by Euclid, B. II. p. 12. $a^2=b^2+x^2+2xy$, $\therefore y=\frac{a^2-b^2-x^2}{2x}=$ (if $m^2=a^2-b^2$) $\frac{m^2-x^2}{2x}$; hence the quantity of light varies as $\frac{b}{x^2}-\frac{m^2-x^2}{2x^3}=\frac{2bx-m^2+x^2}{2x^3}$ which is therefore a maximum; hence it's fluxion $\frac{\overline{2b\dot{x}+2x\dot{x}}\times 2x^3-6x^2\dot{x}\times\overline{2bx-m^2+x^2}}{4x^6}=0$, or it's numerator $4bx^3\dot{x}+4x^4\dot{x}-12bx^3\dot{x}+6m^2x^2\dot{x}-6x^4\dot{x}=0$, or by dividing by $2x^2\dot{x}$ and uniting the like terms, we have $-x^2-4bx+3m^2=0$, $\therefore$ $x^2+4bx=3m^2$, a quadratic, from which $x=-2b+\sqrt{4b^2+3m^2}$. Hence we know the three ſides of the triangle ESV to find the angle E of elongation, which $=39^\circ.\ 44'$.

Ex. 12. Let Q be an object placed beyond the principal focus F of a convex lens, to find it's poſition, when it's diſtance Qq from it's image q is the leaſt poſſible.

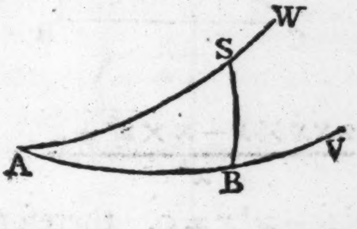
Put $QF=x$, $FE=a$; then (by the Principles of Optics) $x:x+a::x+a:Qq=\frac{\overline{x+a}^2}{x}=$ a min. hence



it's fluxion $\frac{2\dot{x}\times\overline{x+a}\times x-\dot{x}\times\overline{x+a}^2}{x^2}=0$, and by aſſuming the numerator $=0$, and dividing by $x+a$, we have $2x\dot{x}-x\dot{x}-a\dot{x}=0$, or $x-a=0$, $\therefore$ $x=a$.

Ex. 13. To find the Sun's place in the ecliptic, when that part of the equation of time which ariſes from the obliquity of the ecliptic is a maximum.

Let AV be the equator, AW the ecliptic, S the



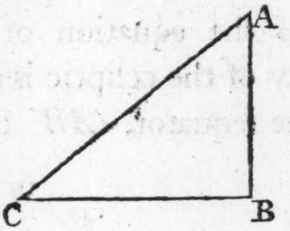
Sun's place, and $SB \perp AV$; then this part of the equation of time is the difference of the Sun's longitude *AS* and right aſcenſion *AB*, turned into time. Put $s=$ coſ. of the angle $A=23°.\ 28'$, $x=$ the tangent of *AS*; then by Spher. Trig. rad. $=1 : s :: x :$ tan. of $AB=sx$; hence by plane trig. the tangent of $\overline{AS-AB}$

$$=\frac{x-sx}{1+sx^2}=\overline{1-s}\times\frac{x}{1+sx^2}=\text{max. or }\frac{x}{1+sx^2}=\text{max.}$$

∴ it's fluxion $\frac{\dot{x}\times\overline{1+sx^2}-2sx\dot{x}\times x}{\overline{1+sx^2}^2}=0$, hence the numerator $\dot{x}+sx^2\dot{x}-2sx^2\dot{x}=0$, ∴ $1-sx^2=0$, and $x=\sqrt{\frac{1}{s}}=1{,}04416$ the tangent of 46°.14′ the Sun's longitude when this part of the equation is a maximum.

Ex. 14. Given the baſe *CB* of an inclined plane *AC*, to find it's altitude *BA*, when the time of the deſcent of a body down the plane is the leaſt poſſible.

Put $a=CB$, $x=BA$, then $\sqrt{a^2+x^2}=AC$; and (by Mechanics) the time down *AC* varies as $\frac{\sqrt{a^2+x^2}}{\sqrt{x}}$, which is therefore a minimum, or $\frac{a^2+x^2}{x}$ is a mini-



mum; hence $\frac{2x\dot{x}\times x-\dot{x}\times\overline{a^2+x^2}}{x^2}=0$, or its numerator $2x^2\dot{x}-a^2\dot{x}-x^2\dot{x}=0$, therefore $x^2=a^2$, and $x=a$.

Ex. 15. Given the baſe *CB*, to find the perpendicular *BA*, ſuch that a body deſcending from *A* to *B*, and then deſcribing *BC* with the velocity acquired, the time through *AB* and *BC* may be the leaſt poſſible.

Put $m = 16\frac{1}{12}$ feet, $a = CB$, $x = BA$; then (by Mechanics) the time down $AB = \sqrt{\frac{x}{m}}$; alſo with the velocity acquired at *B* continued uniform, the body would deſcribe 2 *AB*, or $2x$, in the ſame time; hence, as the ſpace deſcribed with an uniform velocity is as the time, $2x : a :: \sqrt{\frac{x}{m}} : \frac{a}{2x} \times \sqrt{\frac{x}{m}} = \frac{1}{2}a \times \sqrt{\frac{1}{mx}}$ the time of deſcribing *BC*; hence the whole time $= \sqrt{\frac{x}{m}} + \frac{1}{2}a\sqrt{\frac{1}{mx}} = x^{\frac{1}{2}} \times \sqrt{\frac{1}{m}} + \frac{1}{2}ax^{-\frac{1}{2}} \times \sqrt{\frac{1}{m}} =$ a minimum, or $x^{\frac{1}{2}} + \frac{1}{2}ax^{-\frac{1}{2}} = \text{min.}$ $\therefore \frac{1}{2}x^{-\frac{1}{2}}\dot{x} - \frac{1}{4}ax^{-\frac{3}{2}}\dot{x} = 0$, or $x^{-\frac{1}{2}} = \frac{1}{2}ax^{-\frac{3}{2}}$, hence $x = \frac{1}{2}a$.

Ex. 16. Given the baſe *CB* of the inclined plane *AC*, to find it's altitude *BA*, ſuch that the horizontal velocity of a body at *C* deſcending down *AC* may be the greateſt poſſible.

Put $a = CB$, $x = BA$, then $CA = \sqrt{a^2 + x^2}$; now (by Mechanics) the velocity at *C* is as $\sqrt{x}$, and by the reſolution of motion, $\sqrt{a^2 + x^2} : a :: \sqrt{x} : \frac{a\sqrt{x}}{\sqrt{a^2 + x^2}}$ the velocity at *C* in the direction *BC*, which is to be a maximum, or $\frac{x}{a^2 + x^2}$ = a maximum;

$\therefore \dot{x} \times$

$\therefore \frac{\dot{x}\times\overline{a^2+x^2}-2x\dot{x}\times x}{\overline{a^2+x^2}^2}=0$, or the numerator $a^2\dot{x}+x^2\dot{x}-2x^2\dot{x}=0$, hence $x=a$.

Ex. 17. To find when *(A)* $x^3-18x^2+96x-20$ becomes a maximum or minimum.

Aſſume the fluxion $=0$, and $3x^2\dot{x}-36x\dot{x}+96\dot{x}=3\dot{x}\times\overline{x^2-12x+32}=0$, hence $x=4$ or 8. Now to determine which value gives the maximum and which the minimum, find whether the value of the fluxion, juſt before it becomes $=0$, is *poſitive* or *negative*; if *poſitive*, the ſucceeding root gives a *maximum*; if *negative*, a *minimum*; for whilſt a quantity increaſes it's fluxion is poſitive; but when it decreaſes it's fluxion becomes negative, by Art. 16. Now as $3\dot{x}\times\overline{x-4}\times\overline{x-8}=3\dot{x}\times\overline{x^2-12x+32}$; when x is leſs than 4, each factor being negative, the value of the fluxion is poſitive, therefore the root 4 gives *(A)* $x^3-18x^2+96x-20$, a maximum; and as, when x increaſes from 4 to 8, one factor is poſitive and the other negative, the fluxion is negative, therefore the root 8 gives *(A)* a minimum. When we ſay that by making $x=4$ it gives *(A)* a maximum, we mean that *(A)* firſt increaſes till x becomes 4 and then it decreaſes, and not that it is then the greateſt poſſible; for by increaſing x after it exceeds 8, the value of *(A)* increaſes *ſine limite*. And in like manner *(A)* decreaſes whilſt x increaſes from 4 to 8 and then it increaſes, and therefore when $x=8$, *(A)* is ſaid to be a minimum, not that it is then the leaſt poſſible, for if we decreaſe x below 4, *(A)* will decreaſe *ſine limite*.

We have here ſuppoſed x to increaſe; if we ſuppoſe x to decreaſe, and firſt aſſume it greater than 8, then as x decreaſes till it becomes 8, each factor $x-4$, $x-8$ being poſitive the product is poſitive, and therefore it might appear that the root 8 ought to give a maximum; but as x is a *decreaſing* quantity, it's fluxion $(\dot{x})$ is negative by Art. 16; hence $3\dot{x}\times\overline{x-4}\times\overline{x-8}$ is negative till x becomes 8, and therefore this root gives *(A)* a minimum; and whilſt x decreaſes from 8 to 4, $3\dot{x}\times\overline{x-4}\times\overline{x-8}$ is poſitive and therefore 4 gives *(A)* a maximum, agreeable to what was before determined. This inſtance ſhows the neceſſity of attending to the ſigns of the fluxions of increaſing and decreaſing quantities, without which we might have determined *(A)* to have been a maximum when it is a minimum, and a minimum when it is a maximum; for it is merely arbitrary whether we ſuppoſe x to increaſe or decreaſe.

When all the roots of the fluxional equation are impoſſible, as no poſſible value of x can make the equation $=0$, it ſhows that by increaſing x, the given quantity increaſes or decreaſes ſine limite, therefore it admits of no maximum or minimum.

It may happen that the fluxion may be $=0$, and yet the quantity *(A)* may not be a maximum or minimum, which takes place when two of the roots of the fluxional equation are *equal*, becauſe in that caſe, the ſign of the fluxion is the ſame both before and after the equation becomes $=0$ from the ſubſtitution of one of the equal roots. For let the given quantity be $x^4-16x^3+90x^2-216x$, whoſe fluxion is $4x^3\dot{x}-48x^2\dot{x}+180x\dot{x}-216\dot{x}=4\dot{x}\times\overline{x^3-12x^2+45x-54}=4\dot{x}\times\overline{x-3}\times\overline{x-3}\times\overline{x-6}$. Now juſt before $x=3$, this fluxion is negative, and juſt after $x=3$, it is alſo negative;

gative; therefore as the fluxion continues negative whilſt x paſſes through 3, that root does not give (A) a minimum; but as the fluxion paſſes from negative to poſitive whilſt x paſſes from leſs than 6 to more than 6, the root 6 gives (A) a minimum, it's fluxion after that time being poſitive ſhows that (A) then begins to increaſe.

Let the fluxional equation have three equal roots, as in $\dot{x}\times\overline{x-a}\times\overline{x-a}\times\overline{x-a}\times\overline{x-b}$, and let a be leſs than b. Then it is manifeſt that when x is leſs than a, this fluxion is poſitive, and when x paſſes through a and lies between a and b, the fluxion is negative; therefore $x=a$ gives (A) a maximum. Hence it is manifeſt that, in general, when the fluxional equation has an *even* number of equal roots, that root gives (A) neither a maximum nor minimum; but when it has an *odd* number, that root gives (A) either a maximum or minimum. If the Reader wiſh to ſee any thing further on this point, he may conſult LYONS's Fluxions. p. 91.

Ex. 18. To find the value and poſition of the greateſt and leaſt ordinates of a curve, whoſe equation is $y=x^3-px^2+qx-r$, x being the abſciſſa and y the ordinate.

Take the fluxion, and $\dot{y}=3x^2\dot{x}-2px\dot{x}+q\dot{x}$; but when y becomes a maximum, $\dot{y}=0$, hence $3x^2\dot{x}-2px\dot{x}+q\dot{x}=0$; conſequently $x=\frac{p}{3}\pm\sqrt{\frac{p^2}{9}-\frac{q}{3}}$, the values of the abſciſſa correſponding to the required ordinates; and if theſe values of x be reſpectively ſubſtituted into the given equation, the values of the ordinates themſelves will be known. Which of the values of x gives the ordinate a maximum and which a minimum, may be found by Ex. 17. If

$p=18$, $q=60$, $r=10$, then $x=2$ and 10, the two abſciſſæ; which ſubſtituted for x in the given equation, give 46 and -210 for the two ordinates, the latter of which being negative, ſhows that the curve at that point lies below the abſciſſa.

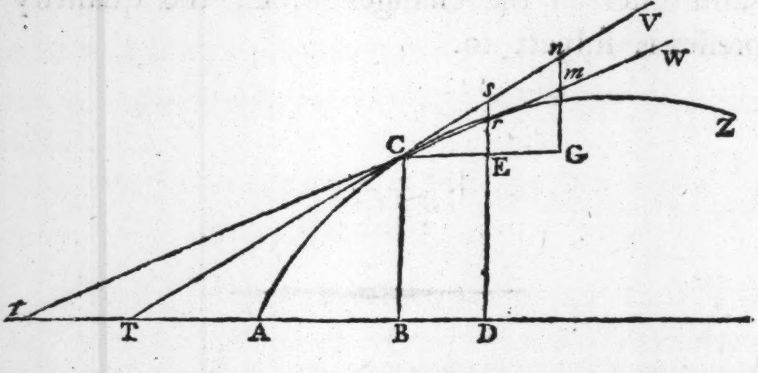
22. In queſtions of a geometrical and philoſophical nature, there are frequently reſtrictions, which enter not into the analytical expreſſion. For inſtance, if a body revolve in an ellipſe and the force tend to the centre, it varies as the diſtance (x) from the center. Now this force in the ellipſe is a maximum at the extremity of the axis major, and a minimum at the extremity of the axis minor; but it is manifeſt that the quantity x is not reſtricted within either of theſe limits, as it may be increaſed or diminiſhed ſine limite; therefore if we put it's fluxion $(\dot{x})=0$, nothing can be determined from that equation reſpecting the points where the force becomes a maximum or minimum. The analytical expreſſion can never be applied, but where it's value is neceſſarily reſtrained by and ſuffer all the changes which the quantity it expreſſes is ſubject to.

TO DRAW TANGENTS TO CURVES.

Prop. X.

Let the curve ACZ be deſcribed by the extremity of the ordinate BC, which moves parallel to itſelf and varies in it's length; to draw a tangent to the curve at any point C.

23. Let TCV be the required tangent; draw any other ordinate Dr and produce it to s; draw alſo CE parallel to BD; join Cr, and produce it to t and W; produce alſo CE to any point G, and draw Gmn parallel to Es. Now let Drs move up to BC, then by the motion of r the line $WrCt$ will revolve about C, and when r coincides with C, it ceaſes to cut the curve between C and Z, and it does not cut it between C and A, for to cut CA, Ct muſt fall below CT, and conſequently CW muſt lie above CV, or r muſt have paſſed s, which it cannot have done, as r has been continually approaching to s and only now coincides with it; therefore when r comes to C, the line Wt, ceaſing to cut the curve, muſt become a tangent, and conſequently WCt will then coincide with VCT. Now whilſt the abſciſſa AB by increaſing becomes AD, the ordinate BC becomes Dr; hence the increment of the ordinate



ordinate BC is Er; and, by ſimilar triangles, the increment CE of the abſciſſa : the cotemporary increment Er of the ordinate :: $CG : Gm$. But when r arrives at C, WC coincides with VC, and conſequently m muſt coincide with n; hence the *limiting* ratio of the increment CE of the abſciſſa to the increment Er of the ordinate, is that of the finite lines $CG : Gn$, which (by ſim. trian.) is the ratio of $CE : Es$, taking DEs in any ſituation before it's coincidence with BC; hence, by Prop. 2. Cor. 1. if CE repreſent the fluxion of the abſciſſa, Es will repreſent the cotemporary fluxion of the ordinate. Put $AB=x$, $BC=y$, then $BD=CE=\dot{x}$, $Es=\dot{y}$; and as BC is parallel to Es, and TB to CE, the angle $TCB=CsE$, and $CTB=sCE$, conſequently the triangles TBC, CEs are ſimilar; hence $\dot{y}\ (Es) : \dot{x}\ (CE) :: y\ (CB) : BT=\frac{y\dot{x}}{\dot{y}}$; therefore ſet off $BT=\frac{y\dot{x}}{\dot{y}}$, join T and C, and TC will be a tangent to the curve at C. If y decreaſe whilſt x increaſes, then $\dot{y}$ becomes negative by Art. 16. and conſequently $\frac{y\dot{x}}{\dot{y}}$, or BT becomes negative, which ſhows that T lies on the other ſide of B. See the Algebra, Art. 474.

The line BT is called the *ſubtangent.*

Examples.

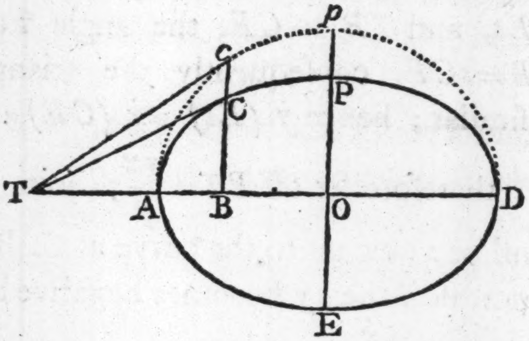
Ex. 1. Let the curve AC be a *parabola*, that is, a curve whoſe abſciſſa varies as any direct power of the ordinate; to draw a tangent to the point C.

The equation expreſſing the relation between x and y is $ax=y^n$, for then $x : y^n :: 1 : a$, a conſtant ratio. Take the fluxion of both ſides of the equation, and we have $a\dot{x}=ny^{n-1}\dot{y}$, hence $\frac{\dot{x}}{\dot{y}}=\frac{ny^{n-1}}{a}$, $\therefore BT=\frac{y\dot{x}}{\dot{y}}=\frac{ny^n}{a}=$

nx, becauſe $\frac{y^n}{a}=x$. If $n=2$, it becomes the common conical parabola, and $BT=2x$.

Ex. 2. To draw a tangent to the *ellipſe ACPDE*, at any point C.

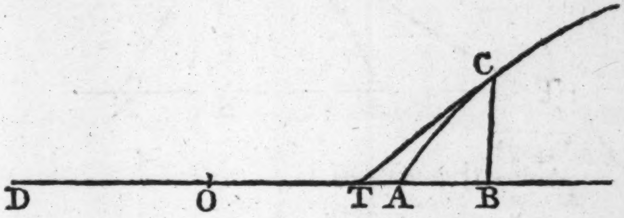
Let AD and PE be the two axes; put $AO=a$, $PO=b$, $AB=x$, $BC=y$, then $BD=2a-x$; and by the property of the ellipſe, $a^2 : b^2 :: \overline{2a-x}\times x : y^2=\frac{b^2}{a^2}\times\overline{2ax-x^2}$; take the fluxions, and $\frac{b^2}{a^2}\times\overline{2a\dot{x}-2x\dot{x}}=2y\dot{y}$, multiply both ſides by $\frac{a^2}{b^2}$, divide by 2 which is common, and alſo by $a-x$, and $\dot{x}=\frac{a^2}{b^2}\times\frac{y\dot{y}}{a-x}$, $\therefore \frac{\dot{x}}{\dot{y}}=\frac{a^2}{b^2}\times\frac{y}{a-x}$; hence $BT=\frac{y\dot{x}}{\dot{y}}=\frac{a^2}{b^2}\times\frac{y^2}{a-x}=\frac{2ax-x^2}{a-x}$, by ſubſtituting $\frac{b^2}{a^2}\times\overline{2ax-x^2}$ for y^2. We may further obſerve, that as this value of TB is independent of b, or PO, if we take $pO=AO$, ſo that ApD may be a circle, and produce BC to c, cT will be a tangent to the circle. If B had been between O and D, ſo that whilſt x increaſes y ſhould decreaſe, then $\dot{y}$ becomes negative by Art. 16. and conſequently $\frac{y\dot{x}}{\dot{y}}$ is negative, which



which ſhows that the ſubtangent BT would lie the other way from B.

Ex. 3. To draw a tangent to the *hyperbola* AC, whoſe major axis is AD.

Biſect AD in O; put $AO=a$, the ſemi-axis minor $=b$, $AB=x$, $BC=y$; then by the property of the hyperbola, $a^2 : b^2 :: \overline{2a+x} \times x : y^2 = \frac{b^2}{a^2} \times \overline{2ax+x^2}$, which is the ſame equation as for the ellipſe, except that the ſigns of x and x^2 are here poſitive; $\therefore BT = \frac{2ax+x^2}{a+x}$.

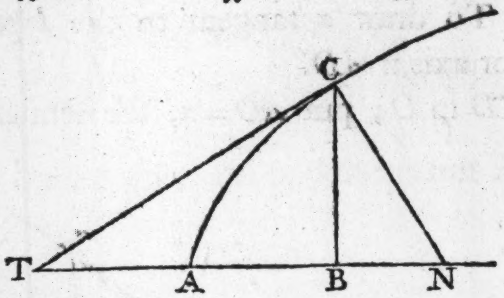


Ex. 4. To draw a tangent to the *Ciſſoid* of *Diocles*, whoſe equation is $y^2 = \frac{x^3}{a-x}$.

Take the fluxion, and $2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + x^3\dot{x}}{\overline{a-x}^2} = \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}^2}$, hence $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}^2}{3ax^2 - 2x^3}$; $\therefore BT = \frac{y\dot{x}}{\dot{y}} = \frac{2y^2 \times \overline{a-x}^2}{3ax^2 - 2x^3} = \frac{2x^3}{a-x} \times \frac{\overline{a-x}^2}{3ax^2-2x^3} = \frac{2x \times \overline{a-x}}{3a-2x}$.

24. Draw CN perpendicular to the tangent, and it is called the *normal*, and NB the *ſub-normal*. Now the triangles TBC, NBC are ſimilar, hence $\frac{y\dot{x}}{\dot{y}}$ $(TB) : y\ (BC) :: y : BN = \frac{y\dot{y}}{\dot{x}}$ the *ſub-normal*; and

$CN^2 = y^2 + \frac{y^2\dot{y}^2}{\dot{x}^2} = y^2 \times \overline{1 + \frac{\dot{y}^2}{\dot{x}^2}} = y^2 \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2}$; hence CN

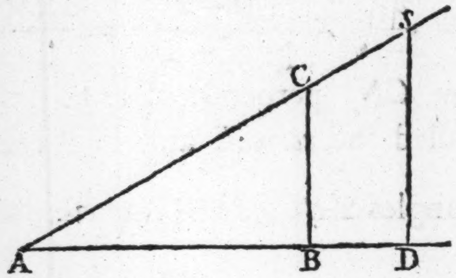


$= y \times \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$ the *normal*.

Ex. Let the curve be a *parabola* whoſe equation is $ax = y^n$; then $a\dot{x} = ny^{n-1}\dot{y}$, and $\frac{\dot{y}}{\dot{x}} = \frac{a}{ny^{n-1}}$, $\therefore BN = \frac{y\dot{y}}{\dot{x}} = \frac{a}{ny^{n-2}}$. In the common parabola, where $n = 2$, $BN = \frac{a}{2}$, a being the latus rectum. Alſo $CN = \sqrt{y^2 + \frac{1}{4}a^2}$.

25. If two quantities begin together and increaſe uniformly, one by x and the other by mx, m being conſtant, then, by the Compoſition of Ratios, the quantities generated will be in the ratio of $x : mx$, or as $1 : m$, a conſtant ratio.

26. If BC move parallel to itſelf, and AB



and BC increaſe uniformly, the locus of the point

C is a ſtraight line. For let *BC* come into the poſition *Ds*; then as *AB* and *BC* begin together and increaſe uniformly, they have always a conſtant ratio to each other, by Art. 25; therefore $AB : BC :: AD : Ds$, which is the property of ſimilar triangles; hence *ACs* is a ſtraight line. Alſo, as *BC* is parallel to *Ds*, $AB : BD :: AC : Cs$, and $AB : AC :: BD : Cs$; but $AB : AC$ in a conſtant ratio; if therefore *BD* the increment of the baſe be conſtant, the cotemporary increment *Cs* of the hypothenuſe muſt be conſtant, or if the former increaſe uniformly, the latter will increaſe uniformly. Hence the two *uniform* motions of *C*, one in a direction parallel to *AB* ariſing from the motion of *BC*, and the other in the direction *BC*, generate an *uniform* motion in a *right* line *AC*.

27. The fluxion of the curve line *AC*, cotemporary to *CE*, *Es* (figure to Art. 23) the fluxions of the abſciſſa and ordinate, is the ſpace that would be deſcribed by the point *C* with it's motion continued uniform for the time in which *CE*, *Es* are deſcribed. Now the motion of *C* ariſes from two motions, one by which it is carried parallel to *AB* by the motion of *BC*, and the other by which it is carried in the direction *BC* by the increaſe of *BC*; therefore the uniform motion of *C* is determined by making theſe two motions become uniform; but when theſe two motions become *uniform*, they are repreſented by *CE* and *Es*, by Art. 23. and theſe two *uniform* motions produce a cotemporary *uniform* motion *Cs*, by Art. 26; hence by Prop. 1. *Cs* will repreſent the cotemporary fluxion of the curve line at the point *C*.

TO DRAW ASYMPTOTES TO CURVES.

DEFINITION.

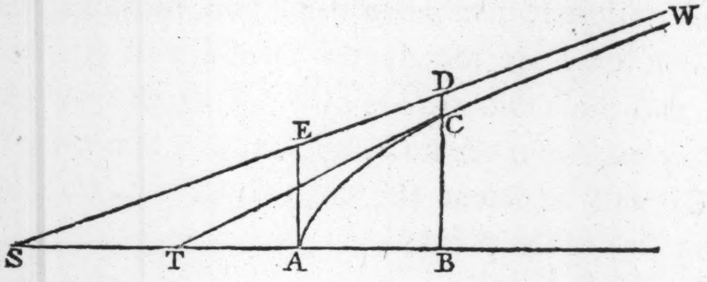
28. If a right line, interſecting the axis of a curve at a finite diſtance, continually approach to the curve, and arrive nearer to it than by any aſſignable diſtance, but indefinitely produced never meets it, it is called an *Aſymptote*.

Prop. XI.

To draw an aſymptote to a curve.

29. Let SDW be an aſymptote to the curve AC; then, by the definition, we may conſider the aſymptote SW as the limit to which the tangent approaches, when the abſciſſa AB is increaſed ſine limite. Draw AE parallel to the ordinate BC produced to D, and let TC be a tangent to the curve at C.

Put $AB=x$, $BC=y$; then by Art. 23. $BT=\frac{y\dot{x}}{\dot{y}}$, hence $AT=\frac{y\dot{x}}{\dot{y}}-x$; from the equation of the curve, find the value of this quantity when x and y are infinite, and if it be then finite, the curve admits of an aſymptote SW, and the value of AS is obtained.



Then having computed the value of BT, find the proportion of TB to BC, and to get their limit, make x

and y infinite, and you get the proportion of SB to BD, becauſe the limits of TB and BC are SB and BD; but, by ſimilar triangles, $SB : BD :: SA : AE$, the ratio therefore of SA to AE is known, and as AS is known, AE is known; therefore the point E is determined; draw SE and produce it indefinitely, and it will be the aſymptote.

Examples.

Ex. 1. Let AC be the common *hyperbola*, AB it's axis; then by Ex. 3. Art. 23. $BT = \frac{2ax+x^2}{a+x}$, therefore $AT = \frac{2ax+x^2}{a+x} - x = \frac{ax}{a+x}$, which, when x is infinite, becomes $\frac{ax}{x} = a = AS$; hence S is the center of the hyperbola. Now $BC = \frac{b}{a} \times \sqrt{2ax+x^2} =$ (when x becomes infinite) $\frac{bx}{a} = BD$; alſo when x becomes infinite, BT or $BS = \frac{2ax+x^2}{a+x} = \frac{x^2}{x} = x$; hence $\frac{bx}{a} : x$ (:: $b : a$) :: $BD : BS :: AE : AS$; but $AS = a$, $\therefore$ $AE = b$; hence draw AE parallel to BC and take it $= b$, join SE and produce it indefinitely, and it will be the aſymptote.

Ex. 2. Let the equation of the curve be $y^3 = ax^2 + x^3$; then $3y^2\dot{y} = 2ax\dot{x} + 3x^2\dot{x}$, and $BT = \frac{y\dot{x}}{\dot{y}} = \frac{3y^3}{2ax+3x^2} = \frac{3ax^2+3x^3}{2ax+3x^2}$, hence $AT = \frac{3ax^2+3x^3}{2ax+3x^2} - x = \frac{ax^2}{2ax+3x^2}$, and when x becomes infinite, this becomes $\frac{a}{3} = AS$; alſo when x becomes infinite, BT or

$BS=x$, and $BC=y=\sqrt[3]{ax^2+x^3}=x=BD$; hence $BS=BD$, conſequently $AE=AS=\frac{a}{3}$; take therefore $AS=\frac{a}{3}$ and $AE=\frac{a}{3}$, join SE and produce it indefinitely, and it will be the aſymptote.

TO DRAW TANGENTS TO SPIRALS.

DEFINITION.

30. If an indefinite right line SM revolve about S, and a point C move in it continually from S, it will deſcribe a curve called a *ſpiral*; S is called the center, and SC it's ordinate.

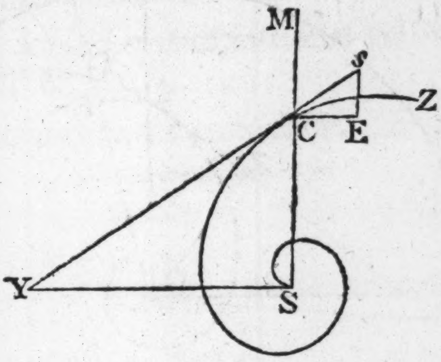
PROP. XII.

To draw a tangent to any point C of the ſpiral.

31. Let YCs be a tangent to the ſpiral at C, and SY perpendicular to SC; draw CE perpendicular, and Es parallel to SM. Now the deſcribing point C has two motions, one in the direction SM, and the other perpendicular to it ariſing from the motion of SM about S. The deſcribing point C is therefore under the very ſame circumſtances as in Art. 23. upon ſuppoſition that CE is there perpendicular to the ordinate CB; the fluxions therefore muſt be repreſented here in like manner as they were there; for the fluxions at the point C in the directions CE, CM and Cs depend (Art.

3.)

3.) entirely upon the velocities of the deſcribing point *C* in thoſe directions, without any regard to what may take place afterwards from the further motion of *MS* about *M*; the fluxions therefore will be juſt the ſame

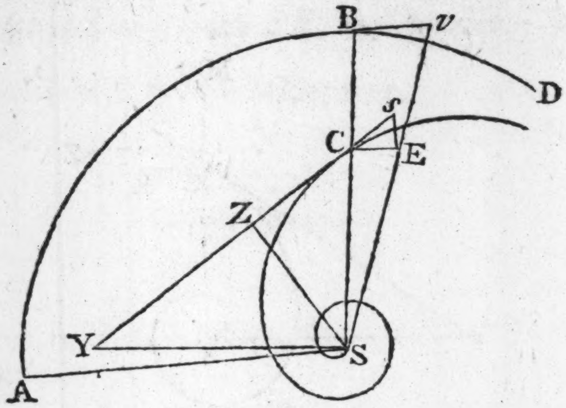


as if the ordinate were moving parallel to itſelf and the deſcribing point *C* had the ſame two motions given to it: hence, by Art. 27. *Cs* is the fluxion of the curve, and by Art. 23. *Es* is the fluxion of the ordinate, and *CE* the fluxion in the direction perpendicular to *SC*. Put $SC=y$, then $Es=\dot{y}$; and, by ſimilar triangles *ECs*, *CSY*, $Es\,(\dot{y}) : CE :: CS\,(y) : SY=\frac{y\times CE}{\dot{y}}$.

Cor. If the point *C* have no motion in the direction *SM*, the curve deſcribed will be a circle, and *Es* becoming $=0$, the cotemporary fluxion of a circular arc whoſe radius *SC* revolves with the ſame angular velocity, will be *CE*.

32. With any radius *SA* deſcribe the circle *ABD*, produce *SC* to *B* and *SE* to *v* meeting *Bv* a tangent to the circle; and ſuppoſe the angle *ASC* to vary as SC^m.

Put $AS=r$, $SC=y$, $AB=x$, $Bv=\dot{x}$ cotemporary with the fluxions *CE*, *Es*; then as x is the meaſure of the angle *ASC*, let us ſuppoſe that when x becomes any quantity s, y becomes t, and $x : s :: y^m : t^m$, $\therefore$ $\frac{sy^m}{t^m} = x$, and $\frac{msy^{m-1}\dot{y}}{t^m} = \dot{x} = Bv$; and by ſimilar triangles SBv, SCE, $r : y :: \frac{msy^{m-1}\dot{y}}{t^m} : CE = \frac{msy^m\dot{y}}{rt^m}$; hence by Art. 27. $SY\left(=\frac{y \times CE}{\dot{y}}\right) = \frac{msy^{m+1}}{rt^m}$.



Cor. Hence, if SZ be perpendicular to CY, we have, by ſim. triangles YSC, SCZ, $CY : CS :: CS : CZ = \frac{CS^2}{CY}$.

Examples.

Ex. 1. If $m = 1$, it becomes the ſpiral of *Archimedes*, and $SY = \frac{sy^2}{rt}$. Hence $CY = \sqrt{\frac{s^2y^4}{r^2t^2} + y^2} = \frac{y\sqrt{y^2+b^2}}{b}$, putting $\frac{r^2t^2}{s^2} = b^2$; $\therefore$ $CZ = \frac{by}{\sqrt{y^2+b^2}}$. Hence alſo $SZ = \frac{y^2}{\sqrt{y^2+b^2}}$.

Ex. 2. If $m = -1$, it becomes the *reciprocal* ſpiral, and $SY = -\frac{st}{r}$, a conſtant quantity.

Ex. 3. If $m=-2$, the ſpiral becomes what Mr. Cotes calls the *lituus*, and $S\Upsilon=\frac{-2st^2}{ry}$.

Ex. 4. The logarithmic ſpiral is generated by the *uniform* angular motion of SC about S, whilſt C recedes from S with a velocity proportional to SC; hence sE, the fluxion of SC, varies as SC; but as the angle CSE is always the ſame in the ſame time, CE will vary as CS; hence $CE : Es :: a : 1$, a conſtant ratio, $\therefore \frac{CE}{Es}=a$, and $S\Upsilon=\frac{y\times CE}{Es}=ay$; conſequently $S\Upsilon : SC :: ay : y :: a : 1$, a conſtant ratio; hence the triangle $SC\Upsilon$ continues always ſimilar to itſelf, and therefore the angle $SC\Upsilon$ is conſtant, and is known from the ratio of $a : 1$.

ON THE BINOMIAL THEOREM.

Prop. XIII.

To expreſs the value of $\overline{a\pm x}|^n$ *by a ſeries.*

33. The ſquare of $1+x$ is $1+2x+x^2$; the cube is $1+3x+3x^2+x^3$; hence it appears, that the coefficients do not depend upon the value of x, but upon the index of the power; therefore if x be diminiſhed and at laſt vaniſh, it will make no alteration in the coefficients. And as by the continual multiplication of $1+x$ we manifeſtly get a quantity with all the powers

of x regularly aſcending, let us aſſume $\overline{1+x}|^{n} = 1 + ax + bx^2 + cx^3 + dx^4 + \&c.$ Now to determine the values of a, b, c, d, &c. take the fluxion of both ſides of this equation, omitting $\dot{x}$ as it will be common to every term; then take the fluxion of the reſulting equation, and ſo on continually, and we get the following equations.

$$n \times \overline{1+x}|^{n-1} = a + 2bx + 3cx^2 + 4dx^3 + \&c.$$

$$n.\overline{n-1} \times \overline{1+x}|^{n-2} = 2b + 2.3cx + 3.4dx^2 + \&c.$$

$$n.\overline{n-1}.\overline{n-2} \times \overline{1+x}|^{n-3} = 2.3c + 2.3.4dx + \&c.$$

&c. &c.

Now make $x=0$, and from the firſt equation, $n=a$; from the ſecond, $n\,\overline{n-1} = 2b$; from the third, $n.\overline{n-1}.\overline{n-2} = 2.3c$, &c. hence $a=n$; $b=n.\frac{n-1}{2}$; $c=n.\frac{n-1}{2}.\frac{n-2}{3}$, &c. where the law of continuation is manifeſt. Hence $\overline{1+x}|^{n} = 1 + nx + n.\frac{n-1}{2}x^2 + n.\frac{n-1}{2}.\frac{n-2}{3}x^3 + \&c.$ Now if n be a whole poſitive number, it is manifeſt that this ſeries will terminate when we come to the coefficient $n.\frac{n-1}{2}\ldots.\frac{n-n}{n+1}$, the laſt factor being $=0$. But the above inveſtigation holds, whether n be a whole number or fraction, poſitive or negative. If n be a negative whole number, the ſeries will never terminate, becauſe the numerators n, $n-1$, $n-2$, &c. become then $-n$, $-n-1$, $-n-2$, &c. and therefore can never become $=0$. Alſo if n be a fraction, it is manifeſt that n, $n-1$, $n-2$, &c. can never become $=0$, becauſe a fraction can never be deſtroyed by the ſubtraction of a whole number from it. Hence the ſeries will always run on ad infinitum, unleſs n be a whole poſitive number. If the binomial be $1-x$, then x becoming negative, the odd powers of x will be negative and the even powers will be poſitive; hence $\overline{1-x}\rvert^{n}=1-nx+n.\frac{n-1}{2}x^{2}-n.\frac{n-1}{2}.\frac{n-2}{3}x^{3}+\&c.$

34. Hence we may expand $\overline{a+x}\rvert^{n}$. For as $a+x=a\times\overline{1+\frac{x}{a}}$, $\therefore\ \overline{a+x}\rvert^{n}=a^{n}\times\overline{1+\frac{x}{a}}\Big\rvert^{n}=$ (by writing $\frac{x}{a}$ for x) $a^{n}\times\overline{1+n.\frac{x}{a}+n.\frac{n-1}{2}.\frac{x^{2}}{a^{2}}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{x^{3}}{a^{3}}+\&c.}$ $=a^{n}+nax^{n-1}+n.\frac{n-1}{2}a^{2}x^{n-2}+n.\frac{n-1}{2}.\frac{n-2}{3}a^{3}x^{n-3}+$ &c. For the different caſes where the ſeries converges or diverges, or becomes $=0$, ſee Dr. WARING's *Med. Anal.* p. 415.

The principal uſe of this rule is to extract the roots of binomials, for if n be a fraction, the ſeries gives that root of the binomial which the fraction expreſſes.

Examples.

Ex. 1. What is the ſquare root of $a^{2}+z^{2}$, or the value of $\overline{a^{2}+z^{2}}\rvert^{\frac{1}{2}}$ in a ſeries?

By the Elements of Algebra, Art. 250. $\overline{a^{2}+z^{2}}\rvert^{\frac{1}{2}}=a\times\overline{1+\frac{z^{2}}{a^{2}}}\Big\rvert^{\frac{1}{2}}$; compare $\overline{1+\frac{z^{2}}{a^{2}}}\Big\rvert^{\frac{1}{2}}$ with $\overline{1+x}\rvert^{n}$, and we have $\frac{z^{2}}{a^{2}}=x$, $\frac{1}{2}=n$; hence by ſubſtitution, $a\times\overline{1+\frac{z^{2}}{a^{2}}}\Big\rvert^{\frac{1}{2}}=$

$a\times 1$

$$a \times \overline{1 + \frac{1}{2} . \frac{z^2}{a^2} + \frac{1}{2} . \frac{\frac{1}{2} - 1}{2} . \frac{z^4}{a^4} + \frac{1}{2} . \frac{\frac{1}{2} - 1}{2} . \frac{\frac{1}{2} - 2}{3} . \frac{z^6}{a^6} + \&c.} = a + \frac{z^2}{2a} - \frac{z^4}{8a^3} + \frac{z^6}{16a^5} - \&c.$$

Ex. 2. What is the fourth root $1 - x$, or the value of $\overline{1 - x}|^{\frac{1}{4}}$ in a ſeries?

Here $n = \frac{1}{4}$, and $\overline{1 - x}|^{\frac{1}{4}} = 1 - \frac{1}{4} x + \frac{1}{4} . \frac{\frac{1}{4} - 1}{2} x^2 - \frac{1}{4} . \frac{\frac{1}{4} - 1}{2} . \frac{\frac{1}{4} - 2}{3} x^3 + \&c. = 1 - \frac{x}{4} - \frac{3}{4.8} x^3 - \frac{3.7}{4.8.12} x^3 - \&c.$

Ex. 3. What is the cube root of $a - z$, or the value of $\overline{a - z}|^{\frac{1}{3}}$ in a ſeries?

Firſt, $\overline{a - z}|^{\frac{1}{3}} = a^{\frac{1}{3}} \times \overline{1 - \frac{z}{a}}|^{\frac{1}{3}}$; and comparing $\overline{1 - \frac{z}{a}}|^{3}$ with $\overline{1 - x}|^{n}$, we have $\frac{z}{a} = x$, $n = \frac{1}{3}$; hence $a^{\frac{1}{3}} \times \overline{1 - \frac{z}{a}}|^{\frac{1}{3}} =$

$$a^{\frac{1}{3}} \times \overline{1 - \frac{1}{3} . \frac{z}{a} + \frac{1}{3} . \frac{\frac{1}{3} - 1}{2} . \frac{z^2}{a^2} - \frac{1}{3} . \frac{\frac{1}{3} - 1}{2} . \frac{\frac{1}{3} - 2}{3} . \frac{z^3}{a^3} + \&c.} = a^{\frac{1}{3}} - \frac{z}{3a^{\frac{2}{3}}} - \frac{z^2}{9a^{\frac{5}{3}}} - \frac{5z^3}{81a^{\frac{8}{3}}} - \&c.$$

Ex. 4. What is the value of $\frac{1}{\sqrt{az - z^2}}$ in an infinite ſeries?

Firſt, $\frac{1}{\sqrt{az - z^2}} = \frac{1}{a^{\frac{1}{2}} z^{\frac{1}{2}} \times \sqrt{1 - \frac{z}{a}}} = \frac{1}{a^{\frac{1}{2}} z^{\frac{1}{2}}} \times \overline{1 - \frac{z}{a}}|^{-\frac{1}{2}}$;

and comparing $\overline{1 - \frac{z}{a}}|^{-\frac{1}{2}}$ with $\overline{1 - x}|^{n}$, we have $\frac{z}{a} = x$, $n = -\frac{1}{2}$; hence $\frac{1}{a^{\frac{1}{2}} z^{\frac{1}{2}}} \times \overline{1 - \frac{z}{a}}|^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}} z^{\frac{1}{2}}} \times$

$\overline{1 -}$

$$1+\frac{-1}{2}.\frac{z}{a}+\frac{-1}{2}.\frac{-\frac{1}{2}-1}{2}.\frac{z^2}{a^2}-\frac{-1}{2}.\frac{-\frac{1}{2}-1}{2}.\frac{-\frac{1}{2}-2}{3}.\frac{z^3}{a^3}+\&c.$$

$$=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}+\frac{z^{\frac{1}{2}}}{2a^{\frac{3}{2}}}+\frac{3z^{\frac{3}{2}}}{8a^{\frac{5}{2}}}+\frac{5z^{\frac{5}{2}}}{16a^{\frac{7}{2}}}+\&c.$$

Ex. 5. To reſolve $\frac{1}{a^2+2ax+x^2}$ into an infinite ſeries.

This quantity is $\frac{1}{\overline{a+x}|^2}=\overline{a+x}|^{-2}$; which compared with $\overline{a+x}|^n$, gives $n=-2$; hence $\overline{a+x}|^{-2}=a^{-2}-2a^{-3}x-2.\frac{-2-1}{2}.a^{-4}x^2-2.\frac{-2-1}{2}.\frac{-2-2}{3}.a^{-5}x^3-\&c.=\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+\&c.$

Ex. 6. What is the value of $\frac{1}{2az+z^2}$ in an infinite ſeries.

This quantity is equal to $\frac{1}{2az\times\overline{1+\frac{z}{2a}}}=\frac{1}{2az}\times\overline{1+\frac{z}{2a}}|^{-1}$; and by comparing $\overline{1+\frac{z}{2a}}|^{-1}$ with $\overline{1+x}|^n$, we have $x=\frac{z}{2a}$, $n=-1$; hence $\frac{1}{2az}\times\overline{1+\frac{z}{2a}}|^{-1}=\frac{1}{2az}\times\overline{1-1.\frac{z}{2a}-1.\frac{-1-1}{2}.\frac{z^2}{4a^2}-\&c.}=\frac{1}{2az}-\frac{1}{4a^2}+\frac{z}{8a^3}-\&c.$

In like manner we muſt proceed in the expanſion and diviſion of all binomial quantities.

35. The value of $\overline{1+x}|^n$ has been aſſumed $=1+ax+$

bx^2+cx^3+&c. and applied in all caſes, whether n is a whole number or a fraction; if n be a whole number, it is manifeſt from the obſervation in Art. 33, that this muſt be the form of the ſeries; but if n be a fraction, it is not ſo obvious that we may aſſume the ſame ſeries; the legality of the aſſumption however in that caſe may be thus ſhown. Let $n=$ any fraction $\frac{r}{s}$, r and s being whole numbers. Now the value of $\overline{1+x}|^{r}$ is expreſſed by $1+ax+bx^2+cx^3+$ &c. but $\overline{1+x}|^{r}$ is the s^{th} power of $\overline{1+x}|^{\frac{r}{s}}$; therefore ſuch a ſeries muſt be aſſumed for $\overline{1+x}|^{\frac{r}{s}}$, that the s^{th} power thereof may give a ſeries of the form $1+ax+bx^2+cx^3+$&c. Now *any* power of the ſeries $1+px+qx^2+rx^3+$&c. will give a ſeries $1+ax+bx^2+cx^3+$&c. therefore we may aſſume a ſeries of that form, where the powers of x regularly aſcend, to repreſent the value of $\overline{1+x}|^{\frac{r}{s}}$.

SECTION III.

ON THE METHOD OF FINDING FLUENTS.

36. THE buſineſs of the *direct* method of fluxions is to find the fluxion from the fluent; to find the fluent from the fluxion is ſometimes called the *inverſe* method of fluxions. It is not difficult to put any quantity into fluxions, there being direct rules for that purpoſe; but there are no direct general rules for finding a fluent from a fluxion, and very often it is impoſſible to do it, as the fluxion may be ſuch as could not ariſe from putting any fluent into fluxions, in which caſe no fluent can be found. We cannot therefore lay down rules for finding the fluents of any other fluxions than thoſe whoſe forms ſhow them to have been derived from ſome fluent.

PROP. XIV.

To find the fluent of any power of a ſimple quantity multiplied by the fluxion of that quantity.

37. The fluxion of x^3 is $3x^2\dot{x}$, therefore we know that the fluent of $3x^2\dot{x}$ is x^3, and it is deduced from the fluxion, by the converſe of the rule for putting x^3 into fluxions. In general, the fluxion of x^n is, by Art. 12. $nx^{n-1}\dot{x}$; therefore the fluent of $nx^{n-1}x$ muſt be x^n, and this fluent is deduced from the fluxion by the following

RULE:

Add unity to the index, divide by the index ſo increaſed, and alſo by the fluxion of the root.

Examples.

Ex. 1. The fluent of $7x^6\dot{x}$ is x^7.

Ex. 2. The fluent of $x^9\dot{x}$ is $\frac{x^{10}}{10}$.

Ex. 3. The fluent of $5x^3\dot{x}$ is $\frac{5x^4}{4}$.

Ex. 4. The fluent of $\frac{7}{9}x^{\frac{5}{3}}\dot{x}$ is $\frac{3}{8}\times\frac{7}{9}\times x^{\frac{8}{3}} = \frac{21}{72}x^{\frac{8}{3}}$.

Ex. 5. The fluent of $\frac{6\dot{x}}{x^9}$ or $6x^{-9}\dot{x}$ is $\frac{6x^{-8}}{-8} = -\frac{3}{4x^8}$.

Ex. 6. The fluent of $\frac{3\dot{y}}{y^{\frac{3}{5}}}$ or $3y^{-\frac{3}{5}}\dot{y}$ is $\frac{5}{2}\times 3y^{\frac{2}{5}} = \frac{15}{2}y^{\frac{2}{5}}$.

38. If $n=0$, or the index of x be -1, the fluxion is $\frac{\dot{x}}{x}$; but this fluxion cannot be generated by x^0, it being (by the Principles of Algebra) unity, a conſtant quantity; hence the fluent of $\frac{\dot{x}}{x}$ cannot be found by this rule.

Prop. XV.

To find the fluent of a compound quantity raiſed to a power, where the term without the vinculum is the fluxion of that under, or in a given ratio to it.

39. The fluxion of $\overline{a^r+x^r}|^n$ is (by Cor. Art. 12.) $n\times \overline{a^r+x^r}|^{n-1}\times rx^{r-1}\dot{x}$, which is found by the ſame rule as the fluxion of x^n. Every complete fluxion therefore of this kind muſt neceſſarily have the index of the unknown quantity without the vinculum, leſs by unity than the index under. Hence every quantity ſo circumſtanced will have it's fluent found by the above rule.

If $r=1$, then $r-1=0$, and $x^0=1$; therefore the fluxion becomes $\overline{a+x}|^{n-1}\times\dot{x}$.

Examples.

Ex. 1. What is the fluent of $\overline{a+x}|^6\times\dot{x}$?

Here the fluxion of the root $a+x$ is $\dot{x}$; hence the fluent is $\dfrac{\overline{a+x}|^7\times\dot{x}}{7\dot{x}}=\dfrac{\overline{a+x}|^7}{7}$.

Ex. 2. What is the fluent of $\overline{a^2+x^2}|^{\frac{1}{2}}\times x\dot{x}$?

Here the fluxion of the root a^2+x^2 is $2x\dot{x}$; hence the fluent is $\dfrac{\overline{a^2+x^2}|^{\frac{3}{2}}\times x\dot{x}}{\frac{3}{2}\times 2x\dot{x}}=\dfrac{\overline{a^2+x^2}|^{\frac{3}{2}}}{3}$.

Ex. 3. What is the fluent of $\overline{a^4-x^4}|^{\frac{5}{3}}\times 3x^3\dot{x}$?

Here the fluxion of the root a^4-x^4 is $-4x^3\dot{x}$; hence the fluent is $\dfrac{\overline{a^4-x^4}|^{\frac{8}{3}}\times 3x^3\dot{x}}{\frac{8}{3}\times -4x^3\dot{x}}=-\dfrac{9\times\overline{a^4-x^4}|^{\frac{8}{3}}}{32}$.

Ex. 4. What is the fluent of $\dfrac{x^8\dot{x}}{\overline{a^9+6x^9}|^{\frac{1}{2}}}$?

This quantity $= \overline{a^9+6x^9}\big)^{-\frac{1}{2}} \times x^8\dot{x}$; and the fluxion of the root a^9+6x^9 is $54x^8\dot{x}$; therefore the fluent is $\frac{\overline{a^9+6x^9}\big)^{\frac{1}{2}} \times x^8\dot{x}}{\frac{1}{2}\times 54x^8\dot{x}} = \frac{\overline{a^9+6x^9}\big)^{\frac{1}{2}}}{27}$.

Quantities which do not at firſt ſtand under this form, may frequently be reduced to it.

Ex. 5. What is the fluent of $\frac{a\dot{x}}{\overline{a^2+x^2}\big)^{\frac{3}{2}}}$.

Firſt, $a^2+x^2 = \overline{\frac{a^2}{x^2}+1} \times x^2 = \overline{a^2x^{-2}+1} \times x^2$; therefore $\overline{a^2+x^2}\big)^{\frac{3}{2}} = \overline{a^2x^{-2}+1}\big)^{\frac{3}{2}} \times x^3$; hence $\frac{a\dot{x}}{\overline{a^2+x^2}\big)^{\frac{3}{2}}} =$ $\frac{a\dot{x}}{\overline{a^2x^{-2}+1}\big)^{\frac{3}{2}} \times x^3} = \overline{a^2x^{-2}+1}\big)^{-\frac{3}{2}} \times ax^{-3}\dot{x}$, where the index of x without is leſs by unity than that under the vinculum; hence the fluent is $\frac{\overline{a^2x^{-2}+1}\big)^{-\frac{1}{2}} \times ax^{-3}\dot{x}}{-\frac{1}{2}\times -2a^2x^{-3}\dot{x}} =$ $\frac{1}{\overline{a^2x^{-2}+1}\big)^{\frac{1}{2}} \times a} = \frac{x}{\overline{a^2+x^2}\big)^{\frac{1}{2}} \times a}$.

No general rule can be given for the reduction of any quantity to it's proper form; in any particular inſtance, it may eaſily be diſcovered, whether it is reducible to it, or not.

40. If both quantities under the vinculum be variable, and the quantity without be the fluxion of the quantity under the vinculum, or in a conſtant ratio to it, the fluent may be found by this rule. Thus, the fluent of $\overline{a^2y^2+y^4}\big)^{\frac{1}{2}} \times \overline{2a^2y\dot{y}+4y^3\dot{y}}$ is $\frac{2}{3} \times \overline{a^2y^2+y^4}\big)^{\frac{3}{2}}$; but theſe caſes ſeldom occur.

PROP. XVI.

To find the fluent of $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$, *where the index of z without the vinculum increaſed by unity, is ſome multiple of the index of z under the vinculum.*

41. Put $a+cz^n=x$, then $z^n=\frac{x-a}{c}$, $\therefore z^{rn}=\frac{\overline{x-a}|^r}{c^r}$, take it's fluxion, and $rnz^{rn-1}\dot{z}=\frac{r\times\overline{x-a}|^{r-1}\dot{x}}{c^r}$, $\therefore z^{rn-1}\dot{z}=\frac{1}{nc^r}\times\overline{x-a}|^{r-1}\times\dot{x}$, hence (putting $r-1=s$) $dz^{rn-1}\dot{z}$ $=\frac{d}{nc^r}\times\overline{x-a}|^s\times\dot{x}=$ (by expanding $\overline{x-a}|^s$) $\frac{d}{nc^r}\times\dot{x}\times$ $\overline{x^s-sax^{s-1}+s.\frac{s-1}{2}a^2x^{s-2}-\&c.}$ ſubſtitute this quantity for $dz^{rn-1}\dot{z}$, and x^m for $\overline{a+cz^n}|^m$, and the given fluxion is transformed to $\frac{d}{nc^r}\times$ $\overline{x^m\dot{x}\times x^s-sax^{s-1}+s.\frac{s-1}{2}a^2x^{s-2}-\&c.}=\frac{d}{nc^r}\times$ $\overline{x^{m+s}\dot{x}-sax^{m+s-1}\dot{x}+s.\frac{s-1}{2}a^2x^{m+s-2}\dot{x}-\&c.}$ the fluent of each of which terms is found by the Rule in Art. 37. hence the fluent required is $\frac{d}{nc^r}\times$ $\overline{\frac{x^{m+s+1}}{m+s+1}-\frac{sax^{m+s}}{m+s}+\frac{s.\frac{s-1}{2}.a^2x^{m+s-1}}{m+s-1}-\&c.}$ Now we muſt conſider when this fluent can be expreſſed in finite terms, and when it fails.

1st If r, and conſequently s, be a whole *poſitive* number, the ſeries ariſing from the expanſion of $x-a$

will terminate, and the fluent can always be found if m be a *poſitive* whole number, or a *poſitive* or *negative* fraction.

2^d If r be a *poſitive* whole number, and m a *negative* whole number greater in magnitude than $s+1$, or r, the fluent can always be found. But if m be a *negative* whole number equal to or leſs in magnitude than r, the denominator of one of the terms muſt become $=0$, in which caſe the fluent of that term fails; for in the fluxion it was of this form $x^{-1}\dot{x}$, which by Art. 38. admits of no fluent by the rule here given; it may however be found by logarithms, as will be explained in Art. 45.

3^d The given fluxion, by reduction, becomes $\overline{az^{-n}+c}|^{m} \times dz^{\overline{m+r}\times n-1}\dot{z}$; hence if m and r be both fractions, but ſuch that $m+r$ may be a whole *negative* number, the fluent can always be found. This will appear, by transforming the fluxion as before; and the ſeries will always terminate; nor can any of the denominators of the terms of the fluent become equal to nothing, ſo as to make the fluent of ſuch term to fail.

TO FIND FLUENTS BY LOGARITHMS.

42. The property of logarithms, or their relation to natural numbers, as has been already explained in Algebra, is this, that as the natural numbers increaſe

in geometric progreſſion, their logarithms increaſe in arithmetic progreſſion.

43. Let a increaſe till it becomes b, c, . . . m, n, o, &c. and ſuppoſe $a : b :: b : c ::$ &c. $:: m : n ::$ &c. then $a : m :: a-b : m-n$; now $a-b$ is the increment of a, and $m-n$ is the increment of m; hence $a : m ::$ the increment of a : the increment of m; and as this is true in every ſtate of the increments, if we make them vaniſh, we have $a : m$ in the *limiting* ratio of the increment of a to the increment of m, that is, as the fluxion of a : the fluxion of m, by Art. 7.

44. Let y be any number, and x it's logarithm; then if x increaſe uniformly, or if $\dot{x}$ be conſtant, y will increaſe in geometric progreſſion, therefore, by the laſt article, y varies as $\dot{y}$, $\therefore \frac{y}{\dot{y}}$ is conſtant; hence $\frac{y\dot{x}}{\dot{y}}$ is conſtant; put therefore $\frac{y\dot{x}}{\dot{y}} = M$, and we have $\dot{x} = M \times \frac{\dot{y}}{y}$; that is, the fluxion of any logarithm is equal to a *conſtant* quantity multiplied into the fluxion of the number divided by the number. The quantity M is called the *modulus* of the ſyſtem, and may be aſſumed of any value.

If $M = 1$, the logarithms are called *hyperbolic*, becauſe the ſame logarithms may be deduced from the hyperbola, as will appear hereafter. In this caſe $\dot{x} = \frac{\dot{y}}{y}$.

Prop. XVII.

To find the fluent of a fluxion, which is the fluxion of any quantity ($\dot{y}$) divided by that quantity (y), or in a given ratio to it.

Put

45. Put $x =$ the hyperbolic logarithm of y, then by Art. 44. $\frac{\dot{y}}{y} = \dot{x}$, and the fluent of $\frac{\dot{y}}{y}$ is x. And as y, although here a ſimple quantity, may repreſent any compound quantity whatever, and $\dot{y}$ it's fluxion, it is manifeſt, that when any fluxional expreſſion appears to be the fluxion of a quantity divided by the quantity itſelf, it's fluent will be the hyperbolic logarithm of that quantity.

Examples.

Ex. 1. The fluent of $\frac{\dot{x}}{x \pm a}$ is the h. l. (hyperbolic logarithm) of $\overline{x \pm a}$.

Ex. 2. The fluent of $\frac{2x\dot{x}}{a^2 + x^2}$ is the h. l. $\overline{a^2 + x^2}$.

Ex. 3. The fluent of $\frac{nx^{n-1}\dot{x}}{a^n + x^n}$ is the h. l. $\overline{a^n + x^n}$.

Theſe fluents are obvious, the given fluxion being manifeſtly the fluxion of the quantity divided by the quantity, for the numerator is the fluxion of the denominator.

Ex. 4. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ is the h. l. of $\overline{x + \sqrt{x^2 \pm a^2}}$.

For put $x^2 \pm a^2 = v^2$, then $x\dot{x} = v\dot{v}$, $\therefore\ x : v :: \dot{v} : \dot{x}$, and $x + v : v :: \dot{x} + \dot{v} : \dot{x}$, hence $\frac{\dot{x} + \dot{v}}{x + v} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, therefore the fluent of $\frac{\dot{x} + \dot{v}}{x + v}$, or of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, is the h. l. $\overline{x + v} =$ h. l. $\overline{x + \sqrt{x^2 \pm a^2}}$.

Ex. 5. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ is the h. l. $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$.

For put $\sqrt{x^2 \pm 2ax} = y$, then $x^2 \pm 2ax + a^2 = y^2 + a^2$, and $x \pm a = \sqrt{y^2 + a^2}$, hence $\dot{x} = \frac{y\dot{y}}{\sqrt{y^2+a^2}}$, conſequently $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}} = \frac{\dot{y}}{\sqrt{y^2+a^2}}$, whoſe fluent, by the laſt article, is h. l. $\overline{y+\sqrt{y^2+a^2}}$ = h. l. $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$

Ex. 6. The fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is the h. l. $\frac{a+x}{a-x}$.

For $\frac{2a\dot{x}}{a^2 - x^2} = \frac{\dot{x}}{a+x} - \frac{-\dot{x}}{a-x}$, whoſe fluent is the h. l. $\overline{a+x}$ − h. l. $\overline{a-x}$ = h. l. $\frac{a+x}{a-x}$, as ſhown in the Algebra, Art. 388. In like manner, the fluent of $\frac{2a\dot{x}}{x^2 - a^2}$ is h. l. $\frac{a-x}{a+x}$.

Ex. 7. The fluent of $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ is the h. l. $\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$.

For put $\sqrt{a^2 \pm x^2} = y$, then $a^2 \pm x^2 = y^2$, $\therefore \pm x\dot{x} = y\dot{y}$, $\therefore \frac{2a\dot{x}}{xy} = \frac{2a\dot{y}}{\pm x^2}$, that is $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}} = \frac{2a\dot{y}}{y^2 - a^2}$, whoſe fluent, by the laſt example, is h. l. $\frac{a-y}{a+y}$ = h. l. $\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$.

Theſe are the moſt uſeful forms of fluxions whoſe fluents can be found by a table of hyperbolic logarithms; which table may be ſupplied, by multiplying the logarithm found from the common tables by 2,30258509, 2,30258509, which will give the correſponding hyperbolic logarithm.

Example: The fluent of $\frac{\dot{x}}{1+x}$ is the h. l. of $\overline{1+x}$; if $x=1$, the fluent is the h. l. of $2=0{,}693147$; if $x=4$, the fluent is the h. l. of $5=1{,}6094379$.

TO FIND FLUENTS BY CIRCULAR ARCS.

Prop. XVIII.

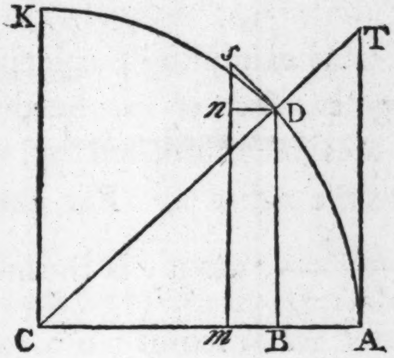
The length of a circular arc for every degree, minute and ſecond, to radius = 1, *being given, to find from thence certain fluents.*

46. Let *AD* be a circular arc whoſe center is *C*, *AT* it's tangent, *DB* it's ſine; draw *ms* parallel to *BD* meeting the tangent *Ds* in *s*, and *Dn* parallel to *Bm*.

Put $CD=a$, $AB=x$, $BD=y$, $AD=z$, $AT=t$, $CT=s$; then by Art. 23. $Ds=\dot{z}$, $Dn=\dot{x}$, $ns=\dot{y}$. Now the triangles *CBD*, *snD* are ſimilar, for they are right angled at *B* and *n*, and the angle $sDn=CDB$, becauſe *nDC* is the complement to each. Hence $y : a :: \dot{x} . \dot{z} = \frac{a\dot{x}}{y}$; but $y=\sqrt{CD^2-CB^2}=\sqrt{a^2-\overline{a-x}^2}=\sqrt{2ax-x^2}$; $\therefore \dot{z}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$. Alſo $\sqrt{a^2-y^2}$ (*BC*)

$: a :: y$

$: a :: \dot{y} : \dot{z} = \frac{a\dot{y}}{\sqrt{a^2-y^2}}$. Again, by ſim. triangles CAT, CBD, $s(CT) : a(CA) :: a(CD) : CB = \frac{a^2}{s}$, $\therefore AB = a - \frac{a^2}{s}$, whoſe fluxion Bm or $Dn = \frac{a^2\dot{s}}{s^2}$; hence, from the ſim. trian. Dsn, CAT, $\sqrt{s^2-a^2}$ $(AT) : s :: \frac{a^2\dot{s}}{s^2} :$ $\dot{z} = \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$. Laſtly, $\sqrt{s^2-a^2} = t$, $\therefore \frac{s\dot{s}}{\sqrt{s^2-a^2}} = \dot{t}$, and $\dot{z}\left(= \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}\right) = \frac{a^2\dot{t}}{s^2} = \frac{a^2\dot{t}}{a^2+t^2}$. Hence the fluxion of the arc AD, or $\dot{z}$, is expreſſed under four different forms in terms of the right ſine, verſed ſine, tangent and ſecant; conſequently the fluent of each of theſe fluxions will be expreſſed by z. Hence



1st Fluent of $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$ is a cir. arc whoſe rad. is a and ſine y.

2d Fluent of $\frac{a\dot{x}}{\sqrt{2ax-x^2}}$ is a cir. arc whoſe rad. is a and verſed ſine x.

3^d Fluent of $\frac{a^2\dot{t}}{a^2+t^2}$ is a cir. arc whoſe rad. is a and tangent t.

4th Fluent of $\frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$ is a cir. arc whoſe rad. is a and ſecant s.

Hence by a table exhibiting the length of circular arcs for all degrees. &c. of the quadrant to radius unity, if theſe arcs be multiplied by a we ſhall have their lengths to the radius a. For example, what is the fluent of $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$, when y is the ſine of 30°? The length of an arc of 30° to radius 1 is 0,5235987; hence the length of the arc to radius a is $a \times 0{,}5235987$, the fluent required. Thus the fluents of all fluxions under any of theſe forms may be found.

47. A fluent can have but one fluxion, but a fluxion may have an infinite number of fluents; thus the fluent of $\dot{x}$ is x, or $x \pm a$, whatever be the value of the conſtant part a. By Prop. 4. in taking the fluxion of a binomial, the conſtant part goes out, and therefore when the fluent is taken back again, that conſtant part does not appear. Now to determine in any particular caſe what this conſtant part is to be, or whether any ſuch a quantity is to be annexed, conſider whether the fluent firſt taken becomes equal to nothing, or of a known value, at the time it ought; if it do, it requires no conſtant quantity to be added; if it do not, ſuch a quantity muſt be annexed to it, as will make it become equal to nothing, or to it's proper value. This is called the *correction* of a fluent.

48. Although the fluxion of a quantity be *relative*,

that is, if $\dot{x}$ denote the fluxion of x, then will $nx^{n-1}\dot{x}$ be the fluxion of x^n, where $\dot{x}$ may be aſſumed of any magnitude, yet the fluents are not at all affected by varying $\dot{x}$, the fluents of theſe quantities $\dot{x}$ and $nx^{n-1}\dot{x}$ being x and x^n, whatever be the value of $\dot{x}$. Hence of whatever magnitude we aſſume the fluxion of any quantity, the fluent will always give the quantity generated. In the following Problems therefore, the fluxion of the area, ſolid, curve line or ſurface, may be aſſumed of any magnitude, and the fluent (corrected if neceſſary) will give the quantity which has been generated.

SECTION

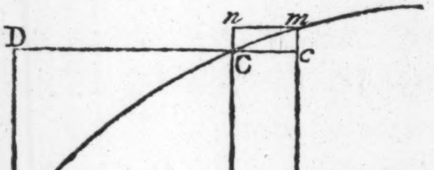
SECTION IV.

TO FIND THE AREAS OF CURVES.

Prop. XIX.

TO find the area ABC of any curve, whoſe ordinate BC is perpendicular to the abſciſſa AB.

49. Let *ABC* be any curvilinear area generated by the uniform motion of the ordinate *BC*; on *AB*, *BC* deſcribe the parallelogram *ABCD*, and conceive this to have been generated by the ſame uniform motion of a line equal and parallel to *AD*; draw *bm* parallel to *BC*, and complete



plete the parallelogram $Bbmn$, and produce DC to c. Then AD being conſtant whilſt BC varies, the next increment of the parallelogram is $BCcb$, and the cotemporary increment of the area ABC is $BCmb$; hence the ratio of the increment $BCcb$ of the parallelogram to the cotemporary increment $BCmb$ of the area ABC, is always nearer to a ratio of equality, than $BCcb:Bnmb$, or nearer than $BC:bm$; now let bm move up to and coincide with BC, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $BC:bm$, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment $BCcb$ of the parallelogram to the cotemporary increment $BCmb$ of the area ABC, is a ratio of equality; therefore by Prop 2. Cor. 1. the fluxion of the parallelogram $ABCD$ is equal to the fluxion of the area ABC; but $BCcb$ being the increment of the parallelogram *uniformly* generated, will repreſent it's fluxion, by Prop. 1. hence the fluxion of the area of the curve ABC will be repreſented by $BCcb$, the cotemporary fluxion of the abſciſſa AB being Bb. Hence if $AB=x$, $BC=y$, $Bb=\dot{x}$, and $A=$ the area ABC, then will $\dot{A}=BCcb=y\dot{x}$; the fluent of which (corrected if neceſſary) gives A.

Cor. Hence the fluxion of any area, generated by the motion of a ſtraight line in a direction perpendicular to itſelf, is as the length of the generating line and it's velocity conjointly. And as a curve line moving in a direction perpendicular to itſelf muſt deſcribe the ſame area as a ſtraight line of the ſame length moving with the ſame velocity, the fluxion of the ſurface generated by a curve line, ſo moving, muſt be as it's length and velocity conjointly.

Examples.

Ex. 1. Let AC be any *parabola*; to find it's area.

Here $ax = y^n$; hence $a\dot{x} = ny^{n-1}\dot{y}$, and $\dot{x} = \frac{ny^{n-1}\dot{y}}{a}$, $\therefore y\dot{x} = \frac{ny^n\dot{y}}{a} = \dot{A}$, whoſe fluent (Ar. 37.) $A = \frac{ny^{n+1}}{\overline{n+1} \times a} + C$ (C being the correction if neceſſary) $= \frac{n}{n+1} \times \frac{y^n}{a} \times y + C = \frac{n}{n+1} \times xy + C$; now when $A = 0$, $x = 0$, $\therefore C = 0$; hence $A = \frac{n}{n+1} \times xy$. If $n = 2$, it becomes the common parabola, and the area $= \frac{2}{3}xy$. If $n = 1$ * the figure becomes a triangle, and the area $= \frac{1}{2}xy$.

Ex. 2. To find the area of a *circle*, whoſe radius is unity.

Let A be the center of the *circle*; draw BC, AP, perpendicular to QR, and join AC. Put $AC = 1$, $AB = x$, $BC = y$; then $x^2 + y^2 = 1$, $\therefore y = \overline{1 - x^2}\big|^{\frac{1}{2}} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} -$ &c. (Ar. 34.); $\therefore \dot{A} = y\dot{x} = \dot{x} - \frac{x^2\dot{x}}{2} - \frac{x^4\dot{x}}{8} - \frac{x^6\dot{x}}{16}$

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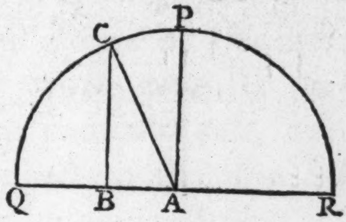
* If $n = 1$, $ax \doteq y$, and $x : y :: 1 : a$, that is, in a conſtant ratio, which is the caſe when AC is a ſtraight line, becauſe the triangle ABC continues always ſimilar to itſelf.

$-\frac{x^6\dot{x}}{16}-\frac{5x^8\dot{x}}{128}-$ &c. the fluxion of the area *BAPC*, whoſe fluent is $A=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}$

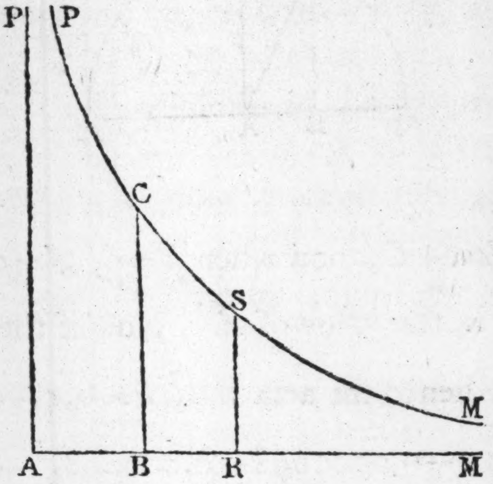


$-\frac{5x^9}{1152}-$ &c.$+C$; now when $x=0$, $A=0$, $\therefore C=0$; hence this is the value of *A*. Now if the arc $PC=30°$, $x=\frac{1}{2}$; hence the area $ABCP=0{,}5-0{,}0208333-0{,}0007812-0{,}0000698-0{,}0000085-0{,}0000012-$ &c. $=0{,}4783055$. Now as $x=\frac{1}{2}$, $y=\sqrt{\frac{3}{4}}$; therefore the area of the triangle $ACB=\frac{1}{4}\times\sqrt{\frac{3}{4}}=0{,}2165063$, which ſubtracted from 0,4783055 leaves 0,2617992 the area of the ſector *ACP*; which multiplied by 12 gives 3,14159 &c. for the area of the whole circle.

Cor. If $r=$ radius of any circle, $a=$ it's area; then, ſince all circles are as the ſquares of their radii, $1^2:r^2::3{,}14159$ &c. $:a=3{,}14159$ &c.$\times r^2$. If $d=$ the diameter, then $r=\frac{d}{2}$, and $r^2=\frac{a^2}{4}$; hence $a=3{,}14159$ &c.$\times\frac{d^2}{4}=0{,}78539$ &c. $\times d^2$.

Ex. 3. To find the area of an *hyperbola* between the aſymptotes AP, AM and the curve.

Put $AB=x$, $BC=y$; then $y=\frac{1}{x^n}$, and the fluxion of the area $APCB=y\dot{x}=\frac{\dot{x}}{x^n}=x^{-n}\dot{x}=\dot{A}$, whoſe



fluent is $A=\frac{x^{1-n}}{1-n}+C$.

Caſe 1. If n be *leſs* than unity, when $A=0$, $x=0$, $\therefore \frac{x^{1-n}}{1-n}=0$, hence $C=0$; therefore the area $APCB$, infinite in extent, is $\frac{x^{1-n}}{1-n}$, a finite quantity when x is finite.

Caſe 2. If n be *greater* than unity, the index $1-n$ being negative, x will come into the denominator, and the fluent will become $A=\frac{1}{\overline{1-n}\times x^{n-1}}+C=-\frac{1}{\overline{n-1}\times x^{n-1}}+C$; now when $A=0$, $x=0$, conſequently

$C=$

$C=\frac{1}{\overline{n-1}\times x^{n-1}}$ is *infinite*, becauſe the denominator becomes $=0$; therefore the area $APCB=\frac{1}{\overline{1-n}\times x^{n-1}}$ $+C$ is *infinite*. Whenever there is a negative index, the quantity muſt always be transferred from the numerator to the denominator, or the contrary, before it's value, in any particular caſe, can be found.

Caſe 3. In reſpect to the area BCM, as this area decreaſes by the ſame quantity that $ABCP$ increaſes, it will have the ſame fluxion, only with a contrary ſign, by Art. 16. hence the fluent will be the ſame with the ſign changed, that is $BCM=\frac{x^{1-n}}{n-1}+C$. If n be *greater* than unity, $BCM=\frac{1}{\overline{n-1}.x^{n-1}}+C$; and when x is infinite, $BCM=0$; hence $0=\frac{1}{\overline{n-1}.x^{n-1}}+C$, and therefore $C=-\frac{1}{\overline{n-1}.x^{n-1}}=0$, x being infinite; conſequently $BCM=\frac{1}{\overline{n-1}.x^{n-1}}$.

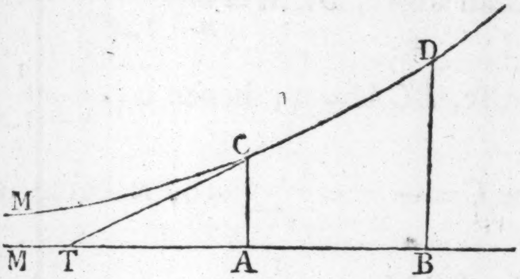
Caſe 4. If n be *leſs* than unity, and x become infinite, $C=\frac{x^{1-n}}{1-n}$, an *infinite* quantity; hence the area $BCM=\frac{x^{1-n}}{n-1}+C$ is *infinite*.

Caſe 5. If $n=1$, this fluent fails (Article 38.), and the hyperbola becomes the common hyperbola Let $AB=BC=1$, $BR=x$, $RS=y$, then AR

$=1+x$, and $y=\frac{1}{1+x}$, therefore the fluxion of the area $BCSR=\frac{\dot{x}}{1+x}$, whoſe fluent, by Art. 45. is the h. l. $\overline{1+x}$, which wants no correction, becauſe when $x=0$, the area $BCRS=0$, and the fluent becomes the h. l. 1 which $=0$. Hence it appears, that any area $BCSR$ is the h. l. of the abſciſſa AR. The *modulus* is here unity.

Ex. 4. Let MCD be the *logarithmic* curve; to find it's area.

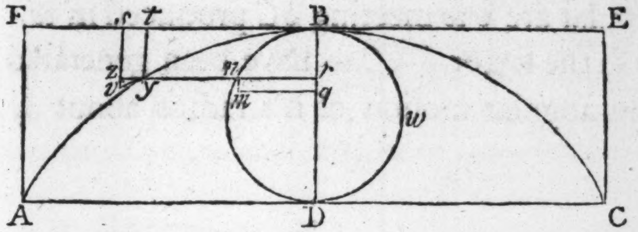
The property of the curve is this, that if the abſciſſa AB increaſe in arithmetical progreſſion, the ordinate BD will increaſe in geometrical progreſſion; $\therefore$ if $x=AB$, $y=BD$, $a=AC$, then by Art. 44. $M=\frac{y\dot{x}}{\dot{y}}$, which (by Article 23.) is the ſubtangent AT; hence $\dot{A}=y\dot{x}=M\dot{y}$, whoſe fluent is $A=My+C$; but when $y=a$, $A=0$, $\therefore 0=Ma+C$, and $C=-Ma$; conſequently $ABDC=My-Ma=AT\times\overline{BD-AC}$. Hence the whole area $DMB=AT\times BD$, becauſe at an infinite diſtance $AC=0$.



Ex. 5. To find the area of the *cycloid* ABC.

Let BD be the axis, on which deſcribe the circle $BnDw$,

$BnDw$, draw $rnyz \perp BD$, and yv a tangent at y, and draw yt, $vs \perp FB$, and vmq parallel to yr, and mn to qr, and join Bn.* Now by the property of the cycloid, the triangles Brn, yzv are ſimilar; hence Br, or ty, $: rn :: zv$, or rq, $: zy$, $\therefore rn \times rq = ty \times zy$, or $\square\ nrqm = \square\ styz$, that is, by Art. 49. the fluxion of the circular area $Bnr =$ the fluxion of the area Bty; and as theſe areas begin together at B, and their cotemporary fluxions are always equal, the quantities generated are equal; hence the area $Bty =$ the circular area Bnr; bring therefore yr down to AD, and we have the whole area $BFA =$ the ſemicircle BnD; hence $BFA + BEC =$ the whole circle $BnDw$. Now the parallelogram $AFEC = AC \times BD =$ (from the nature of the cycloid) circum. $BnDw \times BD =$ (by Art. 51. Ex. 3.) four times the area of the whole circle; hence $ABC =$ three times the whole circle.



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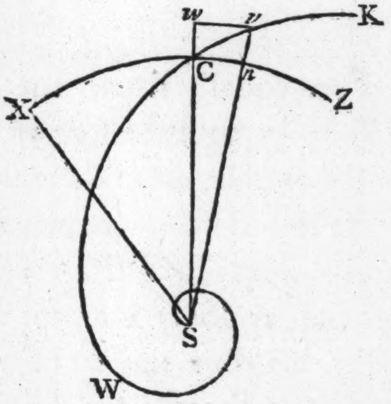
* The Reader is requeſted to ſupply the lines Bn, vm.

TO FIND THE AREAS OF SPIRALS.

Prop. XX.

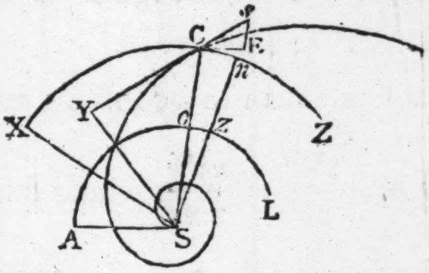
To find the area SWC of a Spiral.

50. Let $SWCK$ be a ſpiral, generated by the uniform angular motion of SC about S; SC any ordinate; with the center S deſcribe the circular arc XCZ; draw any other ordinate Sv, and with the center S deſcribe the circular arc vw meeting SC produced in w. Now conceive the ſector SXC to have been generated by the uniform angular motion of it's radius about S, at the



ſame time that the area SWC of the ſpiral was generated by the ſame uniform angular motion of SC about S. Then SX being conſtant whilſt SC varies, the increment of the ſector SXC is the ſector SCn, and the cotemporary increment of the area SWC of the ſpiral is SCv; hence the ratio of the increment SCn of the ſector SXC to the cotemporary increment SCv of the area SWC, is always nearer to a ratio of equality, than $SCn : Swv$, or nearer than $SC^2 : Sv^2$;* now let Sv move up to and coincide with SC, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $SC^2 : Sv^2$, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment SCn to the increment SCv, is a ratio of equality; therefore by Prop. 2. Cor. 1. the fluxion of the area of the ſector SXC is equal to the fluxion of the area SWC of the ſpiral; but SCn being the increment of the ſector SXC *uniformly* generated, will repreſent it's fluxion, by Prop. 1. hence the fluxion of the area SWC of the ſpiral will be repreſented by CSn.

51. Put $SC=y$, the length of the curve $SWC=z$, $XC=x$, $Cn=\dot{x}$, $A=$ the area SWC; then the ſector $SCn=\frac{y\dot{x}}{2}=\dot{A}$, whoſe fluent is the area SWC. Let sCY be a tangent at C, and SY perpendicular to CY;



draw $CE \perp SC$, and sE parallel to SC; and with the center S and any radius SA deſcribe a circular arc AL. Put $SA=a$, $Ao=w$, $oz=\dot{w}$, $CY=t$, $SY=r$. Then by Art. 31. $Cs=\dot{z}$, $sE=\dot{y}$, $CE=\dot{x}$; and as the triangles CEs, CSY are ſimilar, $t : r :: \dot{y} : \dot{x} = \frac{r\dot{y}}{t}$; hence $SCn = \frac{ry\dot{y}}{2t} = \dot{A}$. Alſo by ſimilar ſectors Soz, SCn, $a : \dot{w} :: y : \dot{x} = \frac{y\dot{w}}{a}$; therefore $SCn = \frac{y^2\dot{w}}{2a} = \dot{A}$. Theſe different expreſſions of the fluxion of the area, are to be uſed as may be convenient.

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* That ſimilar ſectors are as the ſquares of the radii, appears from Euclid, B. XII. p. 2. and B. VI. p. 33.

Examples.

Ex. 1. Let SWC be the *logarithmic* ſpiral; to find it's area.

Here $r : t$ in a conſtant ratio, as $m : n$; hence $\dot{A} = \frac{ry\dot{y}}{2t} = \frac{my\dot{y}}{2n}$, whoſe fluent is $A = \frac{my^2}{4n} + C$; but when $y = 0$, $A = 0$, $\therefore C = 0$; conſequently $A = \frac{my^2}{4n}$.

Ex. 2. Let SWC be the ſpiral of *Archimedes*; to find it's area.

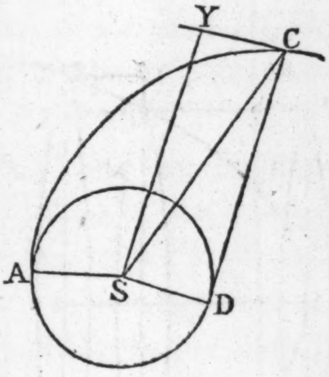
Here $y : w :: m : n$, or in a conſtant ratio; $\therefore \dot{w} = \frac{n\dot{y}}{m}$, conſequently $\dot{A} = \frac{y^2\dot{w}}{2a} = \frac{ny^2\dot{y}}{2ma}$, whoſe fluent is $A = \frac{ny^3}{6ma} + C$; but when $y = 0$, $A = 0$, $\therefore C = 0$; hence $A = \frac{ny^3}{6ma}$.

Ex. 3. Let the ſpiral be a *circle*; to find it's area.

Here y is conſtant, and the fluent of $\dot{A} = \frac{y\dot{x}}{2}$ is $A = \frac{yx}{2}$ the area of the ſector whoſe arc is x; hence if $x =$ the circumference c, the area of the circle $= \frac{cy}{2}$.

Ex. 4. Let AC be the *involute* of the *circle* AD, deſcribed by the extremity C of a ſtring unwinding itſelf from the circle; to find it's area.

It is manifeſt that DC muſt be perpendicular to the curve, or to it's tangent CY, and as SD is alſo $\perp$ to CD, and SY to CY, $SDCY$ is a parallelogram, and $SD=CY=t$, hence $SY=r=\sqrt{y^2-t^2}$; $\therefore\ \dot{A}=\frac{ry\dot{y}}{2t}$ $=\frac{\overline{y^2-t^2}|^{\frac{1}{2}}\times y\dot{y}}{2t}$, whoſe fluent, by Art. 39. is $A=\frac{\overline{y^2-t^2}|^{\frac{3}{2}}}{6t}+C$; but when y (SC) becomes t (SA),



then A, or SAC, is $=0$, and $y^2-t^2=0$, hence $C=0$; $\therefore$ $SAC=\frac{\overline{y^2-t^2}|^{\frac{3}{2}}}{6t}=\frac{DC^3}{6SD}$.

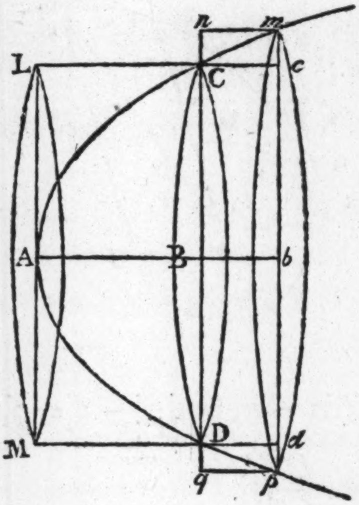
TO

TO FIND THE CONTENT OF SOLIDS.

Prop. XXI.

To find the content of a ſolid generated by the rotation of a curve about it's axis, or by the motion of a plane parallel to itſelf.

52. Let the ſolid *ACD* be conceived to be generated by the uniform motion of the circle *CD*, beginning at *A* and increaſing in magnitude, having it's plane always perpendicular to *AB*, and its center in that line. Circumſcribe this ſolid by the cylinder



MLCD, conceived alſo to be generated at the ſame time by the ſame uniform motion of a circle. Then *AL* being conſtant whilſt *BC* varies, let the circle *CD* move on to *mp*, and the ſolid *CmpD* generated will be the increment of *ACD*; ſuppoſe alſo the circle *CD* to move on to *cd* in the ſame time without increaſing, and it will generate the cotemporary increment $CDdc$ of the cylinder; produce CD to n and p, meeting mn and pq drawn parallel to Bb. Then the ratio of the increment $CDdc$ of the cylinder to the cotemporary increment $CDpm$ of the ſolid ACB, is always nearer to a ratio of equality, than the cylinder $CDdc$: the cylinder $mnqp$, or nearer than $BC^2 : bm^2$. Now let the circle mp move up to and coincide with CD, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $BC^2 : bm^2$, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment $CDdc$ of the cylinder to the cotemporary increment $CDpm$ of the ſolid ACD, is a ratio of equality; therefore by Prop. 2. Cor. 1. the fluxion of the cylinder $MLCD$ is equal to the fluxion of the ſolid ACD; but $CDdc$ being the increment of the cylinder *uniformly* generated, will repreſent it's fluxion, by Prop. 1. hence the fluxion of the ſolid ACD will be repreſented by $CDdc$, the cotemporary fluxion of AB being Bb. Put therefore $x=AB$, $y=BC$, $\dot{x}=Bb$, $S=$ the ſolid ACD, $p=3{,}14159$ &c. then (Art. 49. Ex. 2. Cor.) $py^2=$ the area of the circle CBD; hence the cylinder $CDdc=py^2\dot{x}=\dot{S}$; therefore $S=$ the fluent of $py^2\dot{x}$, corrected if neceſſary.

The ſame reaſoning will manifeſtly hold, if the generating plane be any other figure, and continues always parallel and ſimilar to itſelf. The fluxion therefore of a ſolid thus generated, will be always expreſſed by the area of the generating plane and it's velocity conjointly.

Examples.

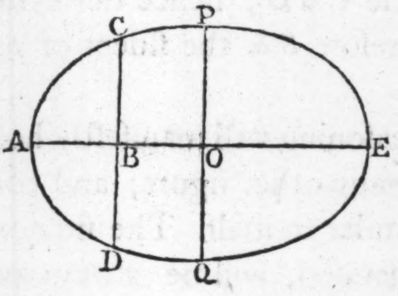
Examples.

Ex. 1. Let *ACD* be a ſolid generated by the revolution of any *parabola* about it's axis.

Here $ax=y^n$; hence $\dot{x}=\frac{ny^{n-1}\dot{y}}{a}$, $\therefore\ \dot{S}=py^2\dot{x}=\frac{npy^{n+1}\dot{y}}{a}$, whoſe fluent is $S=\frac{npy^{n+2}}{\overline{n+2}\times a}+C=\frac{n}{n+2}\times py^2\times\frac{y^n}{a}+C=\frac{n}{n+2}\times py^2x+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence $S=\frac{n}{n+2}\times py^2x$. If $n=2$, the ſolid becomes the common paraboloid, and it's content $=\frac{1}{2}py^2x=\frac{1}{2}$ cylinder *LCDM*. If $n=1$, the curve becomes a ſtraight line, and the ſolid a cone, and it's content $=\frac{1}{3}py^2x=\frac{1}{3}$ cylinder *LCDM*.

Ex. 2. Let *APEQ* be a ſolid generated by the revolution of an *ellipſe* *APEQ* about it's axis *AE*.

Put $AB=x$, $BC=y$, $AO=a$, $PO=b$; then by the



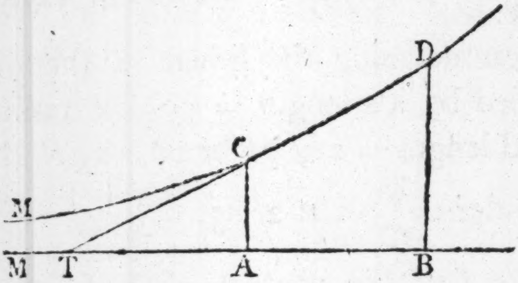
property of the ellipſe, $a^2 : b^2 :: 2ax-x^2 : y^2=\frac{b^2}{a^2}\times\overline{2ax-x^2}$; hence $\dot{S}=py^2\dot{x}=\frac{pb^2}{a^2}\times\overline{2ax\dot{x}-x^2\dot{x}}$, whoſe fluent is $S=\frac{pb^2}{a^2}\times\overline{ax^2-\frac{1}{3}x^3}+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence $S=\frac{pb^2}{a^2}\times\overline{ax^2-\frac{1}{3}x^3}$, which is the ſolid content of ACD; and to get the whole ſolid, we muſt muſt make AB become AE, or make $x=2a$; hence the whole ſolid $=\frac{pb^2}{a^2}\times\overline{4a^3-\frac{8}{3}a^3}=\frac{4pb^2a}{3}$. If the the ellipſe revolve about PQ inſtead of AE, then, as the ſame property of the curve holds for each axis, the ſolid will be $\frac{4pa^2b}{3}$; hence the ſolid generated about AE : ſolid about PQ :: $\frac{4pb^2a}{3}:\frac{4pa^2b}{3}::b:a::PQ:AE$. If $b=a$, the ellipſe $APEQ$ becomes a circle, and the ſolid a *ſphere*, and the content becomes $=\frac{4pb^3}{3}=4{,}18879b^3$. Now the content of a cylinder circumſcribing the ſphere = the area of it's end multiplied by it's length = (as the radius of the end $=b$, and length $=2b$) $pb^2\times 2b=2pb^3$; hence the ſphere : cylinder :: $\frac{4}{3}:2::2:3$.

Ex. 3. To find the content of the ſolid generated by the *Ciſſoid of Diocles* about it's axis.

The equation of this curve is $y^2=\frac{x^3}{a-x}$, (Alg. Art. 426.); hence $\dot{S}=py^2\dot{x}=\frac{px^3\dot{x}}{a-x}=-\frac{px^3\dot{x}}{-x+a}$ = (by diviſion) $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}+\frac{pa^3\dot{x}}{a-x}$; now the fluent of all the terms, except the laſt, is found by Art. 37. and the fluent of the laſt by Art. 45.

45. hence the fluent is $S=-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times-\text{h. l. }\overline{a-x}+C$; now when $x=0$, $S=0$, $\therefore pa^3\times-\text{h. l. }a+C=0$, and $C=pa^3\times\text{h. l. }a$; hence $S=-\frac{1}{3}px^3-\frac{1}{2}pax^2+pax^2+pa^3\times-\text{h.l.}\overline{a-x}+pa^3\times\text{h.l.}a=-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times\text{h. l. }\frac{a}{a-x}$, becauſe h. l. $a-$h. l. $\overline{a-x}=$h. l. $\frac{a}{a-x}$, by the nature of Logarithms, as explained in the Algebra, Art. 388.

Ex. 4. To find the content of the ſolid generated by the *logarithmic* curve *ABDC* revolving about *AB*. Here $y\dot{x}=M\dot{y}$, by Art. 49, Ex.4. $\therefore\ \dot{S}=py^2\dot{x}=Mpy\dot{y}$, whoſe fluent is $S=\frac{Mpy^2}{2}+C$; but when $y=a$, $S=0$,

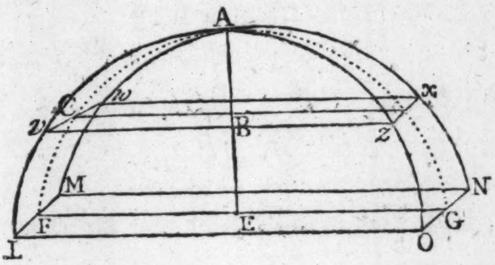


$\therefore\ 0=\frac{Mpa^2}{2}+C$, and $C=-\frac{Mpa^2}{2}$; hence $S=\frac{Mp}{2}\times\overline{y^2-a^2}$.

If $AC=a=0$, then $S=\frac{Mpy^2}{2}=$ the whole ſolid correſponding to the abſciſſa *BM*.

Ex. 5. Let *LAO* be a ſolid called a *Groin*, generated by a ſquare *vwxz* moving parallel to itſelf; and let the ſection FAG through the middle of the oppoſite ſides be a ſemicircle.

Put $a=AE$, $x=AB$, $y=BC$; then by the property of the circle, $y=\sqrt{2ax-x^2}$, therefore the ſide of the ſquare $vwxz=2\sqrt{2ax-x^2}$, hence the area



$vwxz=4\times\overline{2ax-x^2}$, which being the generating plane, it anſwers to py^2 in the other caſes, and therefore $\dot{S}=4\times\overline{2ax\dot{x}-x^2\dot{x}}$, whoſe fluent is $S=4ax^2-\frac{4}{3}x^3+C$; but when $x=0$, $S=0$, $\therefore$ $C=0$; hence $S=4ax^2-\frac{4}{3}x^3$, the ſolid $Avwxz$; and if we make $x=a$, $S=\frac{8a^3}{3}$, the whole ſolid ALN.

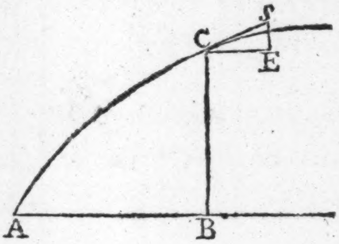
If the ſection FAG be any other figure; or if the two ſections through the two oppoſite ſides be of different figures, the content may be found in like manner.

TO FIND THE LENGTHS OF CURVES.

Prop. XXII.

To find the length of a curve line AC, whoſe ordinate BC is perpendicular to the abſciſſa AB.

53. Put $AB=x$, $BC=y$, $AC=z$; then if Cs be a tangent to the curve, $CE \perp BC$, and $sE \perp CE$, we have, by Art. 27. $CE=\dot{x}$, $sE=\dot{y}$, $Cs=\dot{z}$; and by Euclid, B. I. p. 47. $\dot{z}^2=\dot{x}^2+\dot{y}^2$, $\therefore$ $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, whoſe fluent is z.



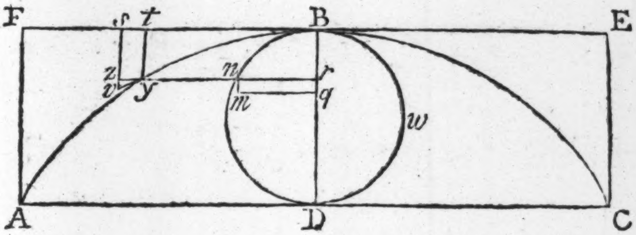
Examples.

Ex. 1. Let AC be a *ſemi-cubical parabola*, whoſe equation is $ax^2=y^3$; to find it's length.

Here $x=\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $\therefore$ $\dot{x}=\frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$, $\therefore$ $\dot{x}^2=\frac{9y\dot{y}^2}{4a}$; hence $\dot{z}^2=\frac{9y\dot{y}^2}{4a}+\dot{y}^2=\overline{\frac{9y}{4a}+1}\times\dot{y}^2=\frac{9y+4a}{4a}\times\dot{y}^2$, $\therefore \dot{z}=\frac{\overline{9y+4a}|^{\frac{1}{2}}\times\dot{y}}{2a^{\frac{1}{2}}}$, whoſe fluent, by Art. 39. is $z=\frac{\overline{9y+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}+C$; now when $y=0$, $z=0$, in which caſe this equation becomes $0=\frac{8a}{27}+C$, ∴ $C=-\frac{8a}{27}$; hence $z=\frac{\overline{9y+4a}\big|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}-\frac{8a}{27}$.

Ex. 2. Let *ByA* be a *cycloid*; to find it's length.

Put $BD=a$, $Br=x$, $By=z$, $yv=\dot{z}$, $yx=rq=\dot{x}$; then by the prop. of the circle, $Br : Bn :: Bn : BD$, ∴ $Bn^2=BD\times Br=ax$, and $Bn=a^{\frac{1}{2}}x^{\frac{1}{2}}$; and by the prop. of the cycloid, $x\ (Br) : a^{\frac{1}{2}}x^{\frac{1}{2}}\ (Bn) :: \dot{x}\ (yx) : \dot{z}\ (vy)$ $=\frac{a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{x}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; hence $z=2a^{\frac{1}{2}}x^{\frac{1}{2}}+C$; but when $x=0$, $z=0$, ∴ $C=0$; conſequently $z=2a^{\frac{1}{2}}x^{\frac{1}{2}}=2Bn$.



Ex. 3. Let *AC* be the common *parabola*; to find it's length.

Here $ax=y^2$, ∴ $\dot{x}=\frac{2y\dot{y}}{a}=$ (if $\frac{a}{2}=b$) $\frac{y\dot{y}}{b}$; hence $\dot{z}^2=\frac{y^2\dot{y}^2}{b^2}+\dot{y}^2=\frac{y^2+b^2}{b^2}\times\dot{y}^2$, ∴ $\dot{z}=\frac{\overline{y^2+b^2}\big|^{\frac{1}{2}}\times\dot{y}}{b}=$ (by multiplying numerator and denominator by $y\times\overline{y^2+b^2}\big|^{\frac{1}{2}}$)

$\frac{1}{b} \times \frac{y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+2b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} + \frac{b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} =$ (by dividing the num. and den. of the laſt term by y) $\frac{1}{2b} \times \overline{y^4+b^2y^2}|^{-\frac{1}{2}} \times 2\overline{y^3\dot{y}+b^2y\dot{y}} + \frac{1}{2}b \times \frac{\dot{y}}{\overline{y^2+b^2}|^{\frac{1}{2}}}$; now the fluent of the firſt term is $\frac{1}{2b} \times \overline{y^4+b^2y^2}|^{\frac{1}{2}}$, by Art. 40. and the fluent of the laſt term is $\frac{1}{2}b \times$ h. l. $\overline{y+\overline{y^2+b^2}|^{\frac{1}{2}}}$, by Art. 45. Ex. 4. hence $z = \frac{1}{2b} \times \overline{y^4+b^2y^2}|^{\frac{1}{2}} + \frac{1}{2}b \times$ h. l. $\overline{y+\overline{y^2+b^2}|^{\frac{1}{2}}} + C$; now when $y=0$, $z=0$, in which caſe the equation is $0=\frac{1}{2}b \times$ h. l. $b+C$; hence $C=-\frac{1}{2}b$ h. l. b; and as the difference of theſe logarithms is, from the nature of logarithms, the h. l. of the former divided by the latter, therefore $z = \frac{1}{2b} \times \overline{y^4+b^2y^2}|^{\frac{1}{2}} + \frac{1}{2}b \times$ h. l. $\frac{y+\overline{y^2+b^2}|^{\frac{1}{2}}}{b}$.

Ex. 4. To find the length of a *circular* arc.

By Art. 46. $\dot{z} = \frac{a^2\dot{t}}{a^2+t^2} =$ (by diviſion) $\dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6} +$&c. hence $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} +$&c. $+C$; but when $t=0$, $z=0$, $\therefore C=0$; hence $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} +$&c. Now if $a=1$, and z be

z be an arc of 30°, then $t = \sqrt{\frac{1}{3}} = 0,5773502$, which being ſubſtituted for t, if we take 12 terms of this ſeries, we get $z = 0,5235987$, the lengh of an arc of 30°; which multiplied by 12 gives 6,2831804 &c. for the length of the circumference of a circle whoſe radius is unity.

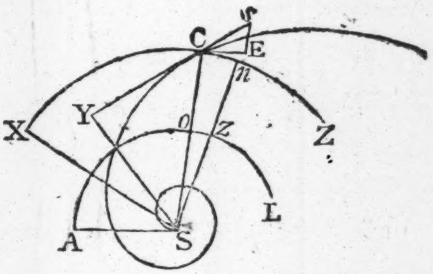
If we take the arc $z = 45°$, then will $t = a$; hence $z = a \times \overline{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}} + \&c.$

TO FIND THE LENGTHS OF SPIRALS.

Prop. XXIII.

To find the length of a ſpiral SC.

54. Let the ordinate $SC = y$, the curve $SC = z$, $CY = t$; then, by Art. 31. $Cs = \dot{z}$, $Es = \dot{y}$; and by ſim.



triangles, $t : y :: \dot{y} : \dot{z} = \frac{y\dot{y}}{t}$, whoſe fluent is z.

Examples

Examples.

Ex. 1. Let SC be the *logarithmic* ſpiral; to find it's length.

Here $t : y :: m : n$, a conſtant ratio; hence $t = \frac{my}{n}$, ∴ $\dot{z} = \frac{n\dot{y}}{m}$, and $z = \frac{ny}{m} + C$; but when $y = 0$, $z = 0$, ∴ $C = 0$; conſequently $z = \frac{ny}{m} = \frac{y^2}{t}$; therefore $CY : CS :: CS :$ the length of the curve.

Ex. 2. Let it be the ſpiral of *Archimedes*; to find it's length.

By Art. 32. Ex. 1. $t = \frac{by}{\sqrt{y^2 + b^2}}$; hence $\dot{z} = \frac{\dot{y}\sqrt{y^2 + b^2}}{b}$, being the ſame as the fluxion of the length of the parabolic arc by Art. 53. Ex. 3. ∴ $z = \frac{1}{2b} \times \overline{y^4 + b^2 y^2}|^{\frac{1}{2}} + \frac{1}{2} b \times$ h. l. $\frac{y + \sqrt{y^2 + b^2}}{b}$.

Ex. 3. Let AC be the *involute* of a *circle*; to find it's length.

Here t is conſtant, by Art. 51. Ex. 4. hence $z = \frac{y^2}{2t} + C$; but when $z = 0$, $y = t$, ∴ $0 = \frac{t^2}{2t} + C$, and $C = -\frac{t^2}{2t}$; hence $z = \frac{y^2 - t^2}{2t} = \frac{SY^2}{2SA}$.

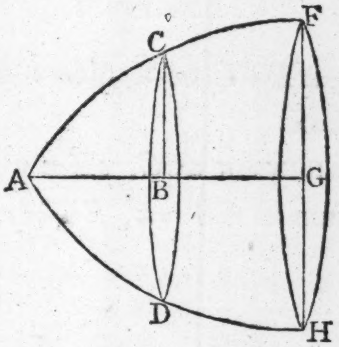
TO

TO FIND THE SURFACES OF SOLIDS.

Prop. XXIV.

To find the ſurface of a ſolid generated by the rotation of a curve about it's axis, or by the motion of a plane parallel to itſelf.

55. Conceiving the ſolid *AFH* to be generated as in Art. 52. by the circle *CD*, the ſurface may be conſidered as generated by the periphery of the circle; the fluxion therefore of the ſurface will be the periphery of the circle multiplied by the velocity with which it flows, by Cor. Art. 49. But the velocity with which any point *C* of the periphery flows is the velocity with which *AC* increaſes at the point *C*, or it is $\dot{z}$, putting $AC = z$. Hence if we put $AB = x$, $BC = y$, $p = 6{,}28318$ &c. the circumference of a circle whoſe radius $= 1$ (Art. 53. Ex. 4.), $S =$ the ſurface *ACD*; then $1 : y :: p : py$ the circumference of the circle *CD*; therefore $\dot{S} = py\dot{z}$ the fluxion of the ſurface; conſequently the fluent of $py\dot{z}$ (corrected if neceſſary) will be the ſurface.



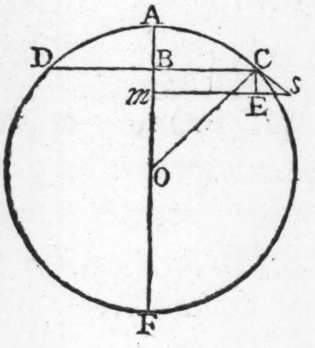
This method of finding the fluxion of the ſurface of a ſolid may be further illuſtrated thus.

Let *ACF* be protended into a ſtraight line, and let an ordinate perpendicular to it, and always equal to the periphery of the circle *CD*, move from *A* to *F* with the ſame velocity as the point *C*, upon the ſolid, moves; then it is manifeſt, that the area generated by this ordinate muſt always be equal to the area generated by the periphery of the circle, the generating lines and their velocities being always equal, and both moving in directions perpendicular to themſelves; hence the fluxion of the ſurface *ACD* = the fluxion of the area of this curve = (by Art. 49.) the ordinate multiplied by the fluxion of the abſciſſa = the periphery of the circle *CD* multiplied by the fluxion of the curve *AC*.

Examples.

Ex. 1. Let *ADFC* be a *ſphere* whoſe center is *O*; to find it's ſurface.

Let *Cs* be a tangent at *C*, *sEm* parallel to *BC*, and *CE* to *Bm*; then if $AB=x$, $BC=y$, $AC=z$, by Art.



23. $Cs=\dot{z}$, $CE=\dot{x}$; and by ſimilar triangles *CEs*, *CBO*, $\dot{z}:\dot{x}::a:y$, $\therefore y\dot{z}=a\dot{x}$; hence $\dot{S}=py\dot{z}=pa\dot{x}$,

the fluxion of the ſurface *DAC*, whoſe fluent $S=pax+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence $S=pax$ the ſurface *DAC*. If we make *AB* equal to *AE*, or $x=2a$, we have $2pa^2$ for the whole ſurface of the ſphere. Now if we conceive *ADFC* to be a great circle of the ſphere, it's area $=\frac{1}{2}pa^2$, by Art. 49.

Ex. 2. Cor. Hence the whole ſurface of a ſphere is equal to four times the area of a great circle of that ſphere.

Cor. As the ſurface $DAC=pax$, it varies as x.

Ex. 2. Let the ſolid *AFH* be generated by the common *parabola*; to find it's ſurface.

Here $ax=y^2$, hence $\dot{x}=\frac{2y\dot{y}}{a}$, and $\dot{x}^2=\frac{4y^2\dot{y}^2}{a^2}$; $\therefore$ (Prop. 22.) $\dot{z}^2=\dot{x}^2+\dot{y}^2=\frac{4y^2\dot{y}^2}{a^2}+\dot{y}^2=\overline{\frac{4y^2}{a^2}+1}\times\dot{y}^2=\frac{4y^2+a^2}{a^2}\times\dot{y}^2$, and $\dot{z}=\frac{\overline{4y^2+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$; hence $\dot{S}=py\dot{z}=\frac{p\times\overline{4y^2+a^2}|^{\frac{1}{2}}\times y\dot{y}}{a}$, whoſe fluent, by Art. 39. is $S=\frac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}+C$; now when $y=0$, $S=0$, in which caſe the equation becomes $0=\frac{pa^2}{12}+C$; hence $C=-\frac{pa^2}{12}$; therefore $S=\frac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}-\frac{pa^2}{12}$.

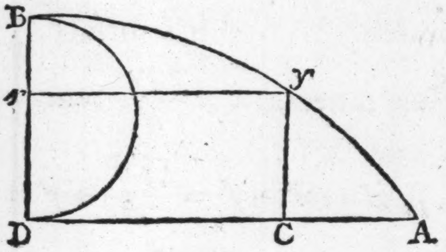
Ex. 3. Let *ALN* be a *groin*, as in Art. 52. Ex. 5. to find it's ſurface.

Put $AB=x$, $BC=y$, $AC=z$; and we have (Art. 46.) $\dot{z}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$; alſo $vw=2BC=2\sqrt{2ax-x^2}$; now vw

is the line generating one of the four ſurfaces; hence $8\sqrt{2ax-x^2}$ anſwers to py in the other caſes; therefore if S be the ſurface Avx, $\dot{S}=8a\dot{x}$, and $S=8ax+C$; but when $x=0$, $S=0$, $\therefore C=0$; conſequently $S=8ax$; and when $x=a$, $S=8a^2$.

Ex. 4. To find the ſurface generated by the *cycloidal curve BA* about it's baſe *DA*.

Put $By=z$, $Br=x$, $rD=yC=y$, $BD=a$; then, by Art. 53. Ex. 2. $\dot{z}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; $\therefore \dot{S}=py\dot{z}=pya^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=p\times$

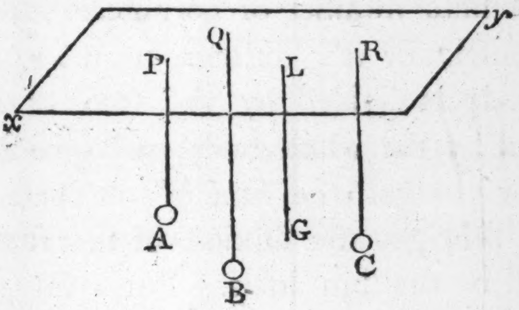


$\overline{a-x}\times a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=pa^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}-pa^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$; hence $S=2pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence $S=2pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}pa^{\frac{1}{2}}x^{\frac{3}{2}}$, the ſurface generated by By; and when $x=a$, we have $S=\frac{4pa^2}{3}$, the whole ſurface generated by BA.

SECTION V.

ON THE CENTER OF GRAVITY.

56. If there be any number of bodies A, B, C, and



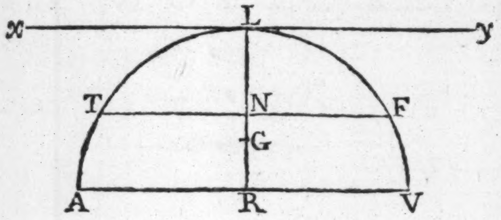
G be their center of gravity; and to any plane xy, perpendiculars AP, BQ, CR, GL be let fall, then (by Mechanics) $LG = \frac{A \times AP + B \times BQ + C \times CR}{A + B + C}$.

Prop. XXV.

To find the center of gravity of a body, conſidered as an area, ſolid, ſurface of a ſolid, or a curve line.

57. Let ALV be any curve, RL the axis, in which the center of gravity muſt lie; for as it biſects every

ordinate *TF* in *N*, the parts on each ſide *LR* will always balance each other, and therefore the body will balance itſelf upon *LR*; conſequently the center of gravity muſt be ſomewhere in that line. Put $LN=x$, $TN=y$, $TL=z$, and draw xy parallel to *TF*; then if we conceive this body to be made up of an indefinite number of corpuſcles, and multiply each corpuſcle by it's diſtance from xy, the ſum of all the products divided by the ſum of all the corpuſcles, or by the whole body, will give *LG* by Art. 56. Now to get the ſum of all theſe products, we muſt firſt get the fluxion of the ſum, and the fluent will be the ſum itſelf. Put $\dot{s}$ for the fluxion of the body at the diſtance x from xy, then will $x\dot{s}$ be the fluxion of the ſum of all the products; alſo $\dot{s}$ is the fluxion of the ſum of all the corpuſcles; therefore by Art. 56. $LG=\frac{\text{flu. } x\dot{s}}{\text{flu. } \dot{s}}$.



1st If the body of an *area*, then $\dot{s}=2y\dot{x}$ by Art. 49. hence $LG=\frac{\text{flu. } 2yx\dot{x}}{\text{flu. } 2y\dot{x}}=\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}}$.

2nd If the body be a *ſolid*, then $py^2\dot{x}=\dot{s}$ by Art. 52. hence $LG=\frac{\text{flu. } py^2x\dot{x}}{\text{flu. } py^2\dot{x}}=\frac{\text{flu. } y^2x\dot{x}}{\text{flu. } y^2\dot{x}}$.

3rd If the body be the *ſurface of a ſolid*, then $\dot{s}=py\dot{z}$ by Art. 55. hence $LG=\frac{\text{flu.}\,pyx\dot{z}}{\text{flu.}\,py\dot{z}}=\frac{\text{flu.}\,yx\dot{z}}{\text{flu.}\,y\dot{z}}$.

4th If the body be a *curve line LT*, then $\dot{s}=2\dot{z}$; hence $LG=\frac{\text{flu.}\,2x\dot{z}}{\text{flu.}\,2\dot{z}}=\frac{\text{flu.}\,x\dot{z}}{\text{flu.}\,\dot{z}}=\frac{\text{flu.}\,x\dot{z}}{z}$.

Examples.

Ex. 1. Let $y=ax^n$ be an equation to any *parabola*; to find it's center of gravity.

As $y=ax^n$, ∴ $yx\dot{x}=ax^{n+1}\dot{x}$, whoſe fluent is $\frac{ax^{n+2}}{n+2}$; alſo $y\dot{x}=ax^n\dot{x}$, whoſe fluent is $\frac{ax^{n+1}}{n+1}$; hence (Art. 57.) $LG=\frac{ax^{n+2}}{n+2}\times\frac{n+1}{ax^{n+1}}=\frac{n+1}{n+2}\times x$.

If $n=\frac{1}{2}$, then $y=ax^{\frac{1}{2}}$, ∴ $y^2=a^2x$, which is the common parabola; hence $LG=\frac{3}{5}x$. If $n=1$, then $y=ax$, and the figure is a triangle; hence $LG=\frac{2}{3}x$.

Ex. 2. Let $y=ax^n$; to find the center of gravity of the ſolid generated by the revolution of this curve about it's axis.

As $y^2=a^2x^{2n}$, ∴ $y^2x\dot{x}=a^2x^{2n+1}\dot{x}$, whoſe fluent is $\frac{a^2x^{2n+2}}{2n+2}$; alſo $y^2\dot{x}=a^2x^{2n}\dot{x}$, whoſe fluent is $\frac{a^2x^{2n+1}}{2n+1}$; hence by Article 57. $LG=\frac{a^2x^{2n+2}}{2n+2}\times\frac{2n+1}{a^2x^{2n+1}}=\frac{2n+1}{2n+2}\times x$.

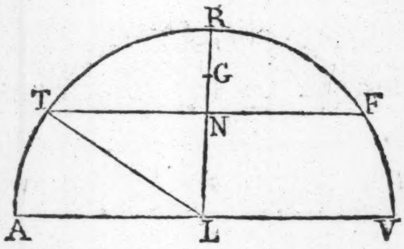
If $n=\frac{1}{2}$, $LG=\frac{2}{3}x$. If $n=1$, the curve becomes a triangle, and the ſolid a cone; and $LG=\frac{3}{4}x$.

Ex. 3. Let *ALV* be a *hemiſpheroid*; to find it's center of gravity.

Put $LR=a$, $AR=b$; then $a^2 : b^2 :: 2ax-x^2 : y^2 = \frac{b^2}{a^2}\times\overline{2ax-x^2}$; hence $y^2x\dot{x}=\frac{b^2}{a^2}\times\overline{2ax^2\dot{x}-x^3\dot{x}}$, whoſe fluent is $\frac{b^2}{a^2}\times\overline{\frac{2}{3}ax^3-\frac{1}{4}x^4}$; alſo $y^2\dot{x}=\frac{b^2}{a^2}\times\overline{2ax\dot{x}-x^2\dot{x}}$, whoſe fluent is $\frac{b^2}{a^2}\times\overline{ax^2-\frac{1}{3}x^3}$; hence by Art. 57. $LG = \frac{\frac{2}{3}ax^3-\frac{1}{4}x^4}{ax^2-\frac{1}{3}x^3}$; and when $x=a$, $LG=\frac{\frac{2}{3}a^4-\frac{1}{4}a^4}{a^3-\frac{1}{3}a^3}=\frac{5a}{8}$ for the whole ſolid. As this is independent of b, if $b=a$, *LG* remains the ſame, and the ſolid becomes an *hemiſphere*.

Ex. 4. Let *ARV* be a *ſemicircle*; to find it's center of gravity.

Put $LN=x$, $TN=y$, $TL=r$; then $x^2+y^2=r^2$, hence



$x\dot{x}+y\dot{y}=0$, ∴ $yx\dot{x}=-y^2\dot{y}$, whoſe fluent is $-\frac{1}{3}y^3+C$, which muſt vaniſh when *TF* coincides with *AV*, or $y=r$;

$y=r$; therefore put r for y, and $-\frac{1}{3}r^3+C=0$, $\therefore C = \frac{1}{3}r^3$; hence the correct fluent of $yx\dot{x}$ is $\frac{1}{3}r^3-\frac{1}{3}y^3$; alſo the fluent of $y\dot{x}$ is (Art. 49.) the area $ATNL$; hence by Art. 57. $LG=\frac{1}{3}\times\frac{r^3-y^3}{ATNL}$; and when $y=0$, $LG= \frac{r^3}{3ARL}$ for the ſemicircle.

Ex. 5. To find the center of gravity of the *arc* ARV.

Put $RN=x$, $NT=y$, $RT=z$; then, (Art. 46.) $\dot{z}:\dot{y}::r:x$, therefore $x\dot{z}=r\dot{y}$, whoſe fluent is ry; hence by Art. 57. $LG=\frac{ry}{z}$; and when $y=r$, $LG =\frac{r^2}{RA}$. The ſame is true for RV; therefore the center of gravity of ARV lies at the ſame diſtance.

Ex. 6. To find the center of gravity of the *ſurface* ARV of an *hemiſphere*.

Here (Art. 46.) we have $\dot{z}:\dot{x}::a:y$, $\therefore y\dot{z}= a\dot{x}$, hence $yx\dot{z}=ax\dot{x}$, whoſe fluent is $\frac{1}{2}ax^2$; alſo the fluent of $y\dot{z}$, or $a\dot{x}$, is ax; hence by Art. 57. $RG =\frac{\frac{1}{2}ax^2}{ax}=\frac{1}{2}x$; and when $x=RL=r$, then $RG= \frac{1}{2}r$ for the hemiſphere.

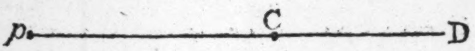
ON

ON THE CENTER OF GYRATION.

DEFINITION.

58. The Center of Gyration is that point of a body revolving about an axis, into which if the whole quantity of matter were collected, the ſame moving force would generate the ſame angular velocity.

59. Let a body p revolve about C, and let a force act at D to oppoſe it's motion. Then the momentum of $p=p\times$ it's velocity $=p\times Cp$, which we may therefore conſider as a power acting at p in



oppoſition to the force at D; but this power acting at the diſtance pC from the center of motion, it's effect to oppoſe a force at D muſt (by the property of the lever) be as $p\times pC^2$. This effect of p to perſevere in it's motion, or which is the ſame, to prevent any change in it's motion, is called it's *inertia*.

PROP. XXVI.

To find the center of gyration of a body.

60. Let a body be conceived to be made up of the particles A, B, C, &c. whoſe diſtances from the axis are a, b, c, &c. and let x be the diſtance of the center of gyration from the axis; then by Art. 59. the inertia of A, B, C, &c. will be $A\times a^2$, $B\times b^2$, $C\times c^2$, &c. and the inertia of all the matter at the diſtance x will

be $\overline{A+B+C+\&c.}\times x^2$; now as the moving force is the ſame in both caſes, the inertia muſt be the ſame when the ſame angular velocity is generated; hence $\overline{A+B+C+\&c.}\times x^2 = A\times a^2 + B\times b^2 + C\times c^2$ +&c. therefore $x = \sqrt{\frac{A\times a^2+B\times b^2+C\times c^2+\&c.}{A+B+C+\&c.}}$; that is, if $\dot{s}$ be the fluxion of the body at the diſtance z from the axis, $x=\sqrt{\frac{\text{flu. } z^2\dot{s}}{s}}$.

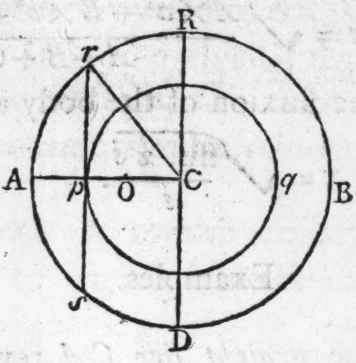
Examples.

Ex. 1. Let the *ſtraight line* CA revolve about C; to find O the center of gyration.

Put $z = Cp$, then $s=z$, and $\dot{s}=\dot{z}$, $\therefore z^2\dot{s}=z^2\dot{z}$, whoſe fluent is $\frac{1}{3}z^3 =$ (when $z=CA$) $\frac{1}{3}CA^3$; hence $CO = \sqrt{\frac{1}{3}CA^2} = CA\sqrt{\frac{1}{3}}$.

Ex. 2. Let a *circle* AB revolve in it's own plane about it's center C; to find it's center of gyration.

Put $p = 6{,}28318$ &c. the circumference of a circle whoſe radius $=1$, $z=Cp$; then the circumference $pq=pz$, and $pz\dot{z}=\dot{s}$; hence the fluent of $z^2\dot{s}$, or of $pz^3\dot{z}$, is $\frac{1}{4}pz^4=$ (when $z=CA=r$) $\frac{1}{4}pr^4$.



Alſo the area of the circle $=\frac{1}{2}pr^2$; hence $CO=\sqrt{\frac{1}{2}r^2}$ $=r\sqrt{\frac{1}{2}}$.

Cor. The ſame muſt be true for a cylinder revolving about it's axis, it being true for every ſection parallel to the end.

Ex. 3. Let *RADB* be a *ſphere* revolving about the diameter *RD*; to find it's center of gyration.

Draw $CA\perp$ and *spr* parallel to *RD*; put $Cr=r$, $Cp=z$, then $pr=\sqrt{r^2-z^2}$; and if $p=6{,}28318$ &c. the ſurface of the cylinder generated by *sr* revolving about *RD* is $pz\times 2\sqrt{r^2-z^2}$; hence $\dot{s}=2pz\dot{z}\sqrt{r^2-z^2}$, and $z^2\dot{s}=2pz^3\dot{z}\sqrt{r^2-z^2}$. Now to find this fluent, put $r^2-z^2=y^2$, then $z^2=r^2-y^2$, and $z^4=r^4-2r^2y^2+y^4$, $\therefore z^3\dot{z}=-r^2y\dot{y}+y^3\dot{y}$; hence $2pz^3\dot{z}\sqrt{r^2-z^2}=2p\times\overline{-r^2y^2\dot{y}+y^4\dot{y}}$, whoſe fluent is $2p\times-\frac{1}{3}r^2y^3+\frac{1}{5}y^5$, and when $z=0$ this fluent ought to vaniſh, but y is then $=r$, and the fluent becomes $2p\times-\frac{2}{15}r^5$; hence the correct fluent is $2p\times\overline{\frac{2}{15}r^5-\frac{1}{3}r^2y^3+\frac{1}{5}y^5}$; and the whole fluent when $z=r$ (in which caſe $y=0$) will be $\frac{4}{15}pr^5$. Now the content of the ſphere $=\frac{2}{3}pr^3$; hence $CO=\sqrt{\frac{2}{5}r^2}=r\sqrt{\frac{2}{5}}$.

ON THE CENTER OF PERCUSSION.

DEFINITION.

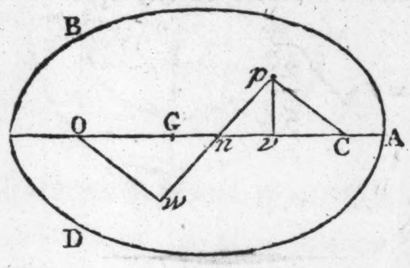
61. The Center of Percuſſion is that point in the axis of a vibrating or revolving body, which ſtriking againſt an immoveable obſtacle, the whole motion is deſtroyed, or the body inclines neither way.

Prop. XXVII.

To find the center of percuſſion of a body.

62. Let ABD be a plane paſſing through the center of gravity G of the body, and perpendicular to the axis of ſuſpenſion which paſſes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis;

then as each particle is thus kept at the ſame diſtance from the axis, the effect, from the rotatory motion about the axis, will not be altered, nor will the center of gravity be changed. Let O be the center of percuſſion, and draw pnw perpendicular to pC, and Ow perpendicular to pw; alſo pv perpendicular to Cn. As the velocity of any particle $p \propto pC$, the momentum of p in the direction $pw \propto p \times pC$, being as the velocity and quantity of matter conjointly; and by the property of the lever, the efficacy of this force to turn the body about O is $p \times pC \times Ow =$ (becauſe $On : Ow :: pC : vC$) $p \times vC \times On = p \times vC \times \overline{CO - Cn} = p \times vC \times CO - p \times vC \times Cn =$ (as $Cn : Cp :: Cp : vC$) $p \times vC \times CO - p \times Cp^2$. Now that the efficacy of all the particles to turn the body about O may be $=0$, we muſt make the *ſum of all the quantities* $p \times vC \times CO -$ *ſum of all the quantities* $p \times Cp^2 = 0$; hence $CO =$



$$\frac{\textit{ſum of all the } p \times Cp^2}{\textit{ſum of all the } p \times vC} = \frac{\textit{ſum of all the } p \times Cp^2}{\textit{body} \times CG},$$ theſe two denominators being equal from the property of the center of gravity, (Art. 56.)

ON

ON THE CENTER OF OSCILLATION.

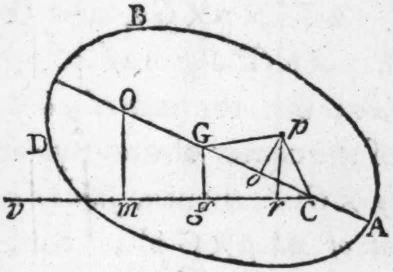
DEFINITION.

63. The Center of Oſcillation is that point in the axis of a vibrating body, at which if a particle were ſuſpended from the axis of motion, it would vibrate in the ſame time the body does.

Prop. XXVIII.

To find the center of oſcillation of any body.

64. Let *ABD* be a body projected upon a plane perpendicular to the axis of rotation, as in Art. 62. the axis paſſing through *C*, and *G* the center of gravity, *O* the center of oſcillation; draw *Cv* parallel to the horizon, *Om*, *Gg*, *pr* perpendicular to it. Then by the property of the lever, the force of gravity to turn the particle *p* about $C \propto p \times Cr$; hence the force of gravity to turn the whole body about $C \propto$ the *ſum*



of all the $p \times Cr$. Alſo the force of gravity to turn a ſingle particle *O* at *O* about $C \propto O \times mC$. Now by Art. 59. the inertia of $p \propto p \times pC^2$; and therefore the inertia of the whole body $\propto$ the *ſum of all the* $p \times pC^2$.

Alſo the inertia of $O \propto O \times OC^2$. Now that the ſame acceleration about C may be produced in the whole body and in the particle O, the moving forces muſt be in proportion to the inertiæ, becauſe, if the power to produce motion be as the power to oppoſe it, the effect muſt be the ſame. Hence, *ſum of all* $p \times Cr : O \times mC$ :: *ſum of all* $p \times Cp^2 : O \times OC^2$, therefore $OC = \frac{\text{ſum of all } p \times Cp^2 \times Cm}{\text{ſum of all } p \times Cr \times OC} = \frac{\text{ſum of all } p \times Cp^2}{\text{body} \times CG}$, becauſe (by ſim. triangles) $Cm : CO :: Cg : CG$, and therefore $\frac{Cm}{CO} = \frac{Cg}{CG}$, and by the property of the center of gravity, *ſum of all* $p \times Cr = \text{body} \times Cg$. Hence the center of oſcillation is the ſame as the center of percuſſion. Or if s be the body, x the diſtance of $\dot{s}$ from the axis of ſuſpenſion, then $CO = \frac{\text{flu. } x^2 \dot{s}}{\text{flu. } x\dot{s}}$, or $= \frac{\text{flu. } x^2 \dot{s}}{s \times CG}$.

65. Join pG, and draw Po perpendicular to CG; then $Cp^2 = CG^2 + Gp^2 - 2CG \times Go$, therefore $p \times Cp^2 = p \times CG^2 + p \times Gp^2 - 2CG \times p \times Go$, and the *ſum of all* $p \times Cp^2 =$ *ſum of all* $p \times CG^2 +$ *ſum of all* $p \times Gp^2 - 2CG \times$ *ſum of all* $p \times Go$; but the *ſum of all* $p \times Go = 0$, from the property of the center of gravity; and the *ſum of all* $p \times CG^2 = \text{body} \times CG^2$; hence, *ſum of all* $p \times Cp^2 = \text{body} \times CG^2 +$ *ſum of all* $p \times Gp^2$; conſequently $CO = \frac{\text{body} \times CG^2 + \text{ſum of all } p \times Gp^2}{\text{body} \times CG} = CG + \frac{\text{ſum of all } p \times Gp^2}{\text{body} \times CG}$; hence $GO = \frac{\text{ſum of all } p \times Gp^2}{\text{body} \times CG}$. Now as the numerator is conſtant, GO varies inverſely as CG; hence if we

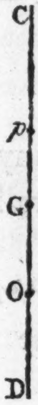
find GO for any one value of CG, we ſhall know every other value of GO.

Cor. If x be the diſtance from C to the center of gyration; then by Art. 60. $x^2 s =$ *ſum of all* $p \times Cp^2$; and by Art. 64. $CO \times s \times CG =$ *ſum of all* $p \times Cp^2$; hence $x^2 = CO \times CG$, and $CG : x :: x : CO$.

Examples.

Ex. 1. Let CD be a *ſtraight line* ſuſpended at C; to find the center of oſcillation.

Put $x = Cp$; then the fluent of $x^2 \dot{s} =$ flu. $x^2 \dot{x} = \frac{1}{3} x^3$

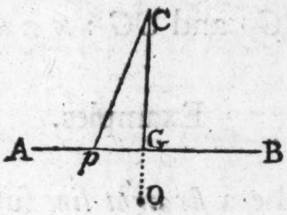


$=$ (when $x = CD$) $\frac{1}{3} CD^3$. Alſo *body* $\times CG = CD \times \frac{1}{2} CD = \frac{1}{2} CD^2$; hence $CO = \frac{2}{3} CD$.

Ex. 2. Let the *line* AB vibrate lengthways in a vertical plane about C, which is equidiſtant from A and B; to find it's center of oſcillation.

Draw CG perpendicular to AB; and put $CG = a$, $Gp = x$; then $pC^2 = a^2 + x^2$; and the fluent of $Cp^2 \times \dot{s}$

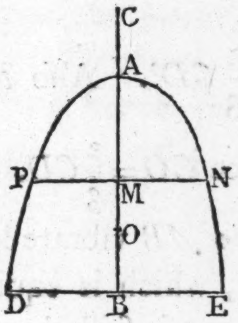
= fluent of $a^2\dot{x} + x^2\dot{x} = a^2x + \frac{1}{3}x^3 =$ (when $x = AG$) $a^2 \times AG + \frac{1}{3}AG^3$; hence for the whole line AB, it be-



comes $2a^2 \times AG + \frac{2}{3}AG^3$. Alſo *body* $\times CG = a \times AB = a \times 2AG$; hence $CO = \frac{2a^2 \times AG + \frac{2}{3}AG^3}{a \times 2AG} = a + \frac{AG^2}{3a}$.

Ex. 3. Let DAE be any *parabola* vibrating *flatways*, or about an axis parallel to PMN; to find the center of oſcillation.

Put $AC = d$, $AM = x$, $PM = y$, then $ax^n = y$; hence $2y\dot{x} = 2ax^n\dot{x} = \dot{s}$; and the fluent of $CM^2 \times \dot{s}$, or



$2.\overline{d+x}^2 \times ax^n\dot{x}$, or $2.d^2ax^n\dot{x} + 4dax^{n+1}\dot{x} + 2ax^{n+2}\dot{x}$, is $\frac{2d^2ax^{n+1}}{n+1}$

$\frac{2d^2ax^{n+1}}{n+1}+\frac{4dax^{n+2}}{n+2}+\frac{2ax^{n+3}}{n+3}$, which vaniſhes when $x=0$, and therefore it wants no correction. Alſo the fluent of $CM\times\dot{s}$, or $\overline{d+x}\times 2ax^n\dot{x}$ is $\frac{2dax^{n+1}}{n+1}+\frac{2ax^{n+2}}{n+2}$; hence if the former be divided by the latter, we get (by reduction) $CO=$

$$\frac{\overline{n+2}.\overline{n+3}.d^2+\overline{n+1}.\overline{n+3}.2dx+\overline{n+1}.\overline{n+2}.x^2}{\overline{n+2}.\overline{n+3}.d+\overline{n+1}.\overline{n+3}.x}.$$

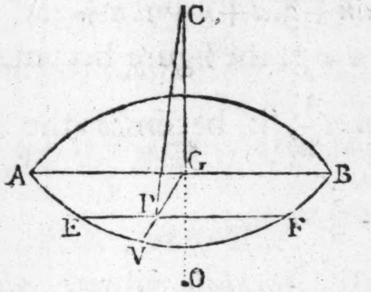
If $d=0$, and $n=1$, the figure becomes a triangle, and $AO=\frac{3}{4}x$. If $n=\frac{1}{2}$, it becomes the common parabola, and $AO=\frac{5}{7}x$.

Ex. 4. Let the *parabola* vibrate *edgeways*, and let it be ſuſpended at A; to find the center of oſcillation.

By Ex. 2. the ſum of the products of each particle of the line PN into the ſquare of it's diſtance from A is $2x^2\times y+\frac{2}{3}y^3=2x^2\times ax^n+\frac{2}{3}a^3x^{3n}$; hence $2ax^{n+2}\dot{x}+\frac{2}{3}a^3x^{3n}\dot{x}$ is the fluxion of the ſum of the products for the whole body; whoſe fluent is $\frac{2ax^{n+3}}{n+3}+\frac{2a^3x^{3n+1}}{3.\overline{3n+1}}$. Alſo the fluent of $AM\times\dot{s}$ is the ſame as before, d being now $=0$; hence $AO=\frac{\overline{n+2}.x}{n+3}+\frac{a^2.\overline{n+2}.x^{2n-1}}{3.\overline{3n+1}}$. If $n=\frac{1}{2}$, $AO=\frac{5x}{7}+\frac{a^2}{3}$. If $n=1$, $AO=\frac{3x}{4}+\frac{a^2x}{4}$ for a triangle. If $a=1$, $AO=x$.

Ex. 5. Let CG be perpendicular to the plane of the circle ABV, and let the *circle* vibrate about an axis paſſing through C and parallel to AB; to find the center of oſcillation.

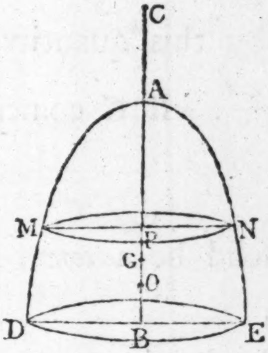
Draw GPV perpendicular to AB, and EF parallel to AB. Put $AG=r$, $CG=a$, $GP=x$, then $CP^2=a^2+x^2$, $PE=\sqrt{r^2-x^2}$, and $EF=2\sqrt{r^2-x^2}$; hence



$EF\times CP^2=\overline{a^2+x^2}\times 2\sqrt{r^2-x^2}$, which multiplied by $\dot{x}$ gives $\overline{a^2\dot{x}+x^2\dot{x}}\times 2\sqrt{r^2-x^2}$ for the fluxion of the ſum of the products of each particle of the area $ABFE$ multiplied into the ſquare of it's diſtance from the axis of vibration. Now to find the fluent, we have the fluent of $a^2\times 2\sqrt{r^2-x^2}\times\dot{x}=a^2\times$ area $ABFE$ by Art. 49. and when $x=r$, the fluent $=a^2$ AVB; and as the ſame is true for the other ſemicircle, the whole fluent is $a^2\times$ circle. The fluent of the ſecond part, $2x^2\dot{x}\sqrt{r^2-x^2}$, may be found thus. Let $\dot{x}\sqrt{r^2-x^2}=\dot{A}$, $x^2\dot{x}\sqrt{r^2-x^2}$, $=\dot{B}$, and $x\times\overline{r^2-x^2}\big)^{\frac{3}{2}}=P$; then by taking the fluxion of the laſt, we have $\dot{P}=\dot{x}\times\overline{r^2-x^2}\big)^{\frac{3}{2}}-3x^2\dot{x}\sqrt{r^2-x^2}=\dot{x}\times\overline{r^2-x^2}\times\sqrt{r^2-x^3}-3x^2\dot{x}\sqrt{r^2-x^2}=r^2\dot{x}\sqrt{r^2-x^2}-4x^2\dot{x}\sqrt{r^2-x^3}$, that is, that is, $\dot{P} = r^2\dot{A} - 4\dot{B}$, hence (by taking the fluents) $P = r^2A - 4B$, and $B = \frac{r^2A - P}{4}$; therefore the fluent of $2x^2\dot{x}\sqrt{r^2 - x^2}$ is $\frac{r^2A - P}{2}$; but when $x = r$, $P = 0$; and the fluent becomes $\frac{r^2A}{2} = \frac{r^2}{8} \times$ circle, becauſe $A = \frac{1}{4}$ of the circle when $x = r$; and for both ſemicircles it becomes $\frac{r^2}{4} \times$ circle; hence the whole fluent is $\overline{a^2 + \frac{1}{4}r^2}$ × circle, which is the ſum of the products of each particle of the circle × the ſquare of it's diſtance from the axis of vibration. Alſo $a \times$ circle = the denominator for the value of CO; hence, by dividing the former by the latter, we get $CO = a + \frac{r^2}{4a}$.

Ex. 6. Let the ſolid formed by the rotation of the curve DAE about it's axis AB, vibrate about C; to find the center of oſcillation.

By Ex. 5. the ſum of the products of each par-



ticle of the circle MN into the ſquare of it's diſtance from the axis $= \overline{CP^2+\frac{1}{4}PN^2}\times$ circle $MN =$ $\overline{CP^2+\frac{1}{4}PN^2}\times p\times PN^2$ (p being $=3.14159$ &c.) $=p\times$ $\overline{CP^2\times PN^2+\frac{1}{4}PN^4}=p\times\overline{d+x}^2\times\overline{y^2+\frac{1}{4}y^4}$; hence $p\dot{x}\times$ $\overline{d+x}^2\times\overline{y^2+\frac{1}{4}y^4}$ is the fluxion of the ſum of all ſuch products for the whole body; the fluent of which divided by $CG\times$ *body*, gives CO.

Ex. 7. Let the ſolid be a *paraboloid*; to find the center of oſcillation.

Here $ax=y^2$; hence $p\dot{x}\times\overline{d+x}^2\times\overline{y^2+\frac{1}{4}y^4}$ is equal to $p\dot{x}\times\overline{d+x}^2\times\overline{ax+\frac{1}{4}a^2x^2}$, whoſe fluent is $\frac{1}{2}pad^2x^2+$ $\frac{2}{3}padx^3+\frac{1}{4}pax^4+\frac{1}{12}pa^2x^3$; alſo (Art. 52. Ex. 1.) the body $=\frac{1}{2}pax^2$; and (Art. 57. Ex. 2.) $AG=\frac{2}{3}x$, $\therefore CG=d+\frac{2}{3}x$; hence $CG\times body=\frac{1}{2}padx^2+\frac{1}{3}pax^3$; dividing therefore the above fluent by this quantity, we have $CO=$ $\frac{6d^2+8dx+3x^2+ax}{6d+4x}$. If C coincide with A, $d=0$, and $CO=\frac{3x+a}{4}$.

Ex. 8. Let the ſolid be a *cone*; to find the center of oſcillation.

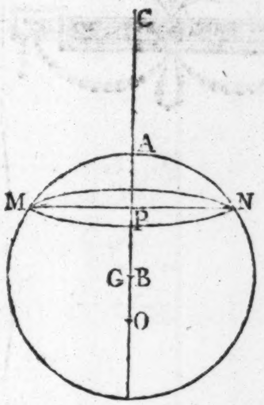
Put $AB=a$, $BD=b$; then $a:b::x:y=\frac{bx}{a}=$ (if $m=\frac{b}{a}$) mx; hence $p\dot{x}\times\overline{d+x}^2\times\overline{y^2+\frac{1}{4}y^4}=p\dot{x}\times$

$\overline{d+x}^2$

$\overline{\overline{d+x}^2\times m^2x^2+\frac{1}{4}m^4x^4}$, whoſe fluent is $\frac{1}{3}pd^2m^2x^3+\frac{1}{2}pdm^2x^4+\frac{1}{5}pm^2x^5+\frac{1}{20}pm^4x^5$; alſo (Art. 52. Ex. 1.) the body $=\frac{1}{3}pm^2x^3$; and (Art. 57. Ex. 2.) $AG=\frac{3}{4}x$, $\therefore CG=d+\frac{3}{4}x$; hence $CO=\frac{20d^2+30dx+12x^2+3m^2x^2}{20d+15x}=$ $\frac{20d^2+30da+12a^2+3b^2}{20d+15a}$ for the whole cone, when $x=a$, and $mx=y=b$.

Ex. 9. Let the body be a *ſphere*; to find the center of oſcillation.

Let B be the center; then if $BA=r$, $y^2=2rx-x^2$. In this caſe, it will be moſt convenient to apply the rule in Art. 65. that is, to get the value of CO when C coincides with A, and from thence to deduce it's value in any other caſe. Now when C coincides with A, $d=0$, and the expreſſion becomes $p\dot{x}\times\overline{x^2y^2+\frac{1}{4}y^4}=p\times$



$\overline{r^2x^2\dot{x}+rx^3\dot{x}-\frac{3}{4}x^4\dot{x}}$, whoſe fluent is $\frac{1}{3}pr^2x^3+\frac{1}{4}prx^4$ –

$-\frac{3}{20}px^5$; and when $x=2r$ it becomes $\frac{28}{15}pr^5$ for the whole ſphere. Alſo the *body* $\times CG$ (*G* now coinciding with *B*) $=\frac{4}{3}pr^3\times r=\frac{4}{3}pr^4$; therefore $AO=1\frac{2}{5}r$; conſequently $BO=\frac{2}{5}r$. Hence (Art. 65.) if $d=CB$, $d:r::\frac{2}{5}r:\frac{2r^2}{5d}=BO$ when the point of ſuſpenſion is at *C*; therefore $CO=d+\frac{2r^2}{5d}$.

SECTION

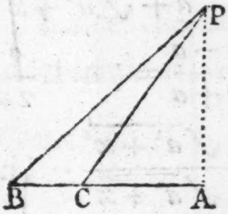
SECTION VI.

ON THE ATTRACTION OF BODIES.

Prop. XXIX.

TO determine the attraction of a corpuſcle P towards a right line BA, in the direction PA perpendicular to AB, ſuppoſing the attraction to each particle of the line to vary inverſely as the ſquare of the diſtance.

66. Put $PA=a$, $AC=x$, then $PC^2=a^2+x^2$, and therefore the attraction of P towards a particle at C is as $\frac{1}{a^2+x^2}$; and by the reſolution of forces $\sqrt{a^2+x^2} : a$



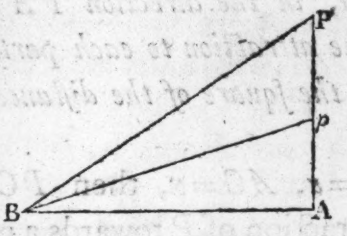
$:: \frac{1}{a^2+x^2} : \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ the attraction in the direction PA;

hence $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the fluxion of the whole force, whoſe fluent (Art. 39. Ex. 5.) is $\frac{x}{\overline{a^2+x^2}|^{\frac{1}{2}}\times a}$, which, when $x = AB$, becomes $\frac{AB}{PB \times PA}$ for the whole attraction.

PROP. XXX.

If the line PA be perpendicular to BA, to find the attraction of PA to BA, upon the ſame law of force.

67. Put $a = AB$, $x = Ap$, then (Art. 66.) the attraction of a corpuſcle at p to $AB = \frac{a}{x\sqrt{a^2+x^2}}$; hence $\frac{a\dot{x}}{x\sqrt{a^2+x^2}}$ is the fluxion of the attraction required; whoſe fluent (Art. 45. Ex. 7.) is $\frac{1}{2}$ h. l. $\frac{a-\sqrt{a^2+x^2}}{a+\sqrt{a^2+x^2}}$; now when $x = 0$, this becomes $\frac{1}{2}$ h. l. $\frac{a-a}{a+a} = \frac{1}{2}$ h. l. $\frac{0}{2a}$; hence the fluent corrected is $\frac{1}{2}$ h. l. $\frac{a-\sqrt{a^2+x^2}}{a+\sqrt{a^2+x^2}} - \frac{1}{2}$ h. l. $\frac{0}{2a} =$ (when $x = AP$) $\frac{1}{2}$ h. l. $\frac{AB-BP}{AB+BP} - \frac{1}{2}$ h. l. $\frac{0}{2AB}$, an infinite quantity.

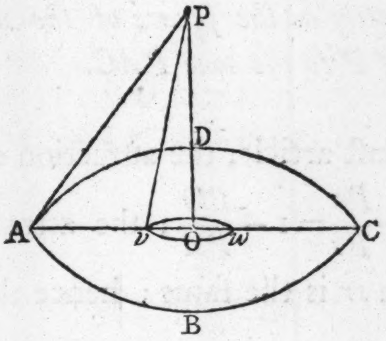


PROP.

Prop. XXXI.

Let O be the center of a circle ABCD, and a corpuſcle P be ſituated in the line OP perpendicular to it's plane; to find the attraction of P to the circle, ſuppoſing the attractive force of P to every particle of the circle to vary as the n^{th} power of the diſtance.

68. Put $PO=a$, $Pv=x$, $p=3{,}14159$, &c. then $Ov^2=x^2-a^2$, and by Art. 49. $p\times\overline{x^2-a^2}=$ the area of the circle vw; hence $2px\dot{x}$ is the fluxion of the area at the diſtance Ov from the center; and by the reſolution of forces, $x : a :: x^n$ (the attraction of P towards v) : ax^{n-1} the attraction of P to a corpuſcle at v in the direction PO; hence the fluxion of the attraction of P towards the circle is as $2px\dot{x}\times ax^{n-1}=2pax^n\dot{x}$, or as $ax^n\dot{x}$, whoſe fluent is $\frac{ax^{n+1}}{n+1}$; but when $x=a$, $Ov=0$, and conſequently the attraction vaniſhes; but in this caſe, the fluent is $\frac{a^{n+2}}{n+1}$; therefore the fluent corrected becomes $\frac{ax^{n+1}}{n+1}-\frac{a^{n+2}}{n+1}$;

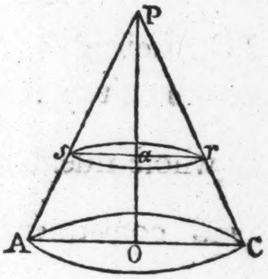


and when $x=PA$ (neglecting the conſtant denominator) it becomes $PO\times PA^{n+1}-PO^{n+2}$, which is as the whole attraction towards the circle. If $n=-2$, it becomes $1-\frac{PO}{PA}$, the denominator neglected being now $=-1$. If n be a negative number greater than 1, and the radius AO become infinite, ſo that PA becomes infinite, then PA being in the denominator, the firſt term $PO\times PA^{n+1}=0$, and the attraction is as PO^{n+2}. Hence if $n=-2$, the attraction becomes unity; therefore the attraction is the ſame at all diſtances PO.

PROP. XXXII.

Let the attractive force of a corpuſcle at P to each particle vary inverſely as the ſquare of the diſtance; to find the attraction of P to the cone PAC.

69. By the laſt article, the attraction of P to the circle sr is as $1-\frac{Pa}{Ps}=1-\frac{PO}{PA}$; the attraction therefore to every ſection sr is the ſame; hence the attraction to

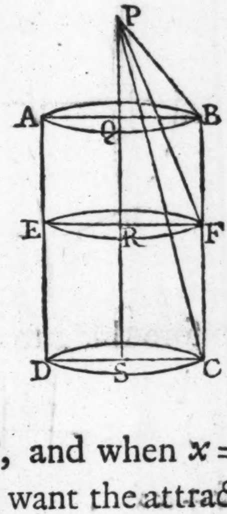


the whole cone is as $\overline{1-\frac{PO}{PA}}\times$ number of ſections, or as $\overline{1-\frac{PO}{PA}}\times PO$ or $PO-\frac{PO^2}{PA}$. Hence for ſimilar cones, $\frac{PO}{PA}$ being conſtant, the attraction varies as the lengths.

Prop. XXXIII.

If a corpuſcle be ſituated at P in the axis SQ of a cylinder, to find it's attraction to the cylinder, ſuppoſing the attractive force to each particle to vary inverſely as the ſquare of the diſtance.

70. Put $RF=a$, $PR=x$, then $PF=\sqrt{a^2+x^2}$; and by Art. 68. the attraction of P towards the circle EF is as $1-\frac{x}{\sqrt{a^2+x^2}}$; hence the fluxion of the attractive force is as $\dot{x}-\frac{x\dot{x}}{\sqrt{a^2+x^2}}$, whoſe fluent is $x-\sqrt{a^2+x^2}$ (Art. 39.); now when $x=PQ$ this fluent



becomes $PQ-PB$, and when $x=PS$, it becomes $PS-PC$; and as we want the attraction of P to the ſolid between theſe two values of x, their difference $SQ+PB-PC$ is as the attraction required.

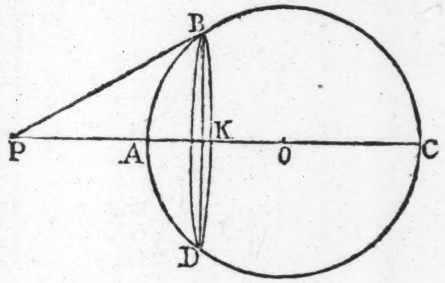
If the length be infinite, then $PC=PS$; therefore $SQ-PC=SQ-PS=-PQ$, and the attraction becomes as $PB-PQ$.

If the diameter AB be infinite, then $PC=PB$; hence the attraction becomes as SQ.

Prop. XXXIV.

To find the attraction of a corpuſcle P to a ſphere, when the attraction to each particle varies inverſely as the ſquare of the diſtance.

71. Let PAC be perpendicular to BD; put the radius $AO=a$, $OP=b$, $AP=b-a=c$, $PK=y$, and let $PB=c+x$, then $AK=y-c$, $CK=2a-y+c$, $\therefore$ $\overline{y-c}\times\overline{2a-y+c}=BK^2=BP^2-PK^2=\overline{c+x}^2-y^2$, hence $y=\frac{2ac+2c^2+2cx+x^2}{2a+2c}=$ (as $b=a+c$)



$\frac{2bc+2cx+x^2}{2b}$; therefore the attraction of P to the circle BD is (Art. 68.) as $1-\frac{2bc+2cx+x^2}{2b\times\overline{c+x}}$, or as $\frac{2ax-x^2}{b\times\overline{c+x}}$; alſo $\dot{y}=\frac{c\dot{x}+x\dot{x}}{b}$; hence the fluxion of the attraction to the ſphere is as $\frac{2ax\dot{x}-x^2\dot{x}}{b^2}$, whoſe fluent is $\frac{ax^2 - \frac{1}{3}x^3}{b^2}$, the attraction to *ABD*; and when $x = 2a$, it is $\frac{4a^3}{3b^2}$, the attraction to the whole ſphere, which therefore varies as $\frac{a^3}{b^2}$.

If the denſity d of the ſphere ſhould vary, then the attraction will vary as $\frac{da^3}{b^2}$.

If the corpuſcle be at the ſurface of the ſphere, then $a = b$, and the attraction varies as da.

Since the quantity of matter m varies as da^3, the attraction varies as $\frac{m}{b^2}$; now if the ſphere were evaneſcent in magnitude, with the ſame quantity of matter, the attraction would be the ſame. Hence the attraction of a corpuſcle to a ſphere is juſt the ſame as if all the matter of ſphere were collected into it's center.

SECTION VII.

ON THE MOTION OF BODIES ATTRACTED TO A CENTER OF FORCE.

Prop. XXXV.

To find the time and velocity of a body deſcending or aſcending in a non-reſiſting medium, in a right line to or from a center of force; ſuppoſing the force to vary as any power of the diſtance from the center.

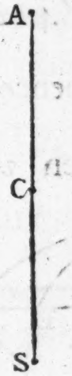
72. Let v be the velocity of the body at any time, x the correſponding ſpace, either that deſcribed, or to be deſcribed, $m = 16\frac{1}{12}$ feet, $t =$ the time, F the force compared with gravity which is repreſented by unity; then $v\dot{v} = \pm 2mF\dot{x}$, the ſign being $+$ when v and x increaſe together, and $-$ when v increaſes as x decreaſes. For by Mechanics, $\dot{v} \propto F \times \dot{t}$, and $\dot{t} \propto \frac{\dot{x}}{v}$, hence $\dot{v} \propto F \times \frac{\dot{x}}{v}$, and $v\dot{v} \propto F \times \dot{x}$, that is $v\dot{v}$ is to $F\dot{x}$ in ſome conſtant ratio; let $v\dot{v} = dF\dot{x}$. Now when a body falls upon the Earth's ſurface, $v^2 = 4mx$

by Mechanics, x being the ſpace deſcribed; hence $v\dot{v} = 2m\dot{x}$; but if x be the ſpace to be deſcribed, and a the whole ſpace, then $v^2 = 4m \times \overline{a-x}$, and $v\dot{v} = -2m\dot{x}$; hence $v\dot{v} = \pm 2m\dot{x}$; but in this caſe $F=1$; therefore $d = \pm 2m$; hence $v\dot{v} = \pm 2mF\dot{x}$. Alſo the velocity of a body moving uniformly is meaſured by the ſpace deſcribed in 1″; therefore for the time $\dot{t}$ correſponding to the ſpace $\pm\dot{x}$, we have $v : 1'' :: \pm\dot{x} : \dot{t} = \pm\frac{\dot{x}}{v}$, becauſe v is the velocity with which $\dot{x}$ is deſcribed in the time $\dot{t}$, and, when the velocity is uniform, the ſpace is as the time.

PROP. XXXVI.

Let a body begin to fall from any point A towards the center of force S; to find the velocity at C, and the time of deſcribing AC.

73. Put $a = SA$, $x = SC$, $v =$ velocity at C, and let the force vary as x^n, and at any diſtance c from S let

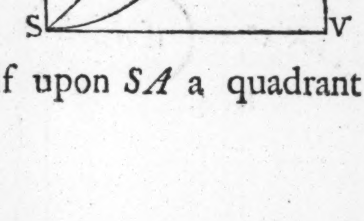


e repreſent the force compared with gravity or unity; then $c^n : x^n :: e : \frac{e}{c^n} \times x^n = \left(\text{if } d = \frac{e}{c^n}\right) dx^n$, the force at the diſtance x; hence $v\dot{v} = -2mdx^n\dot{x}$, and $\frac{v^2}{2} = -\frac{2md}{n+1}\times x^{n+1} + C$; but when $v=0$, $x=a$, and $0 = -\frac{2md}{n+1}\times a^{n+1} + C$, $\therefore C = \frac{2md}{n+1}\times a^{n+1}$; conſequently $\frac{v^2}{2} = \frac{2md}{n+1}\times\overline{a^{n+1}-x^{n+1}}$, and $v = \sqrt{\frac{4md}{n+1}}\times\sqrt{a^{n+1}-x^{n+1}}$. Hence $\dot{t} = -\frac{\dot{x}}{\sqrt{\frac{4md}{n+1}}\times\sqrt{a^{n+1}-x^{n+1}}}$, whoſe fluent gives t; but this can be found only in particular caſes.

Examples.

Ex. 1. If $n=0$, then $x^n=1$, and the force is conſtant, and $v = \sqrt{4md}\times\sqrt{a-x}$. Alſo, $\dot{t} = \frac{-\dot{x}}{\sqrt{4md}\times\sqrt{a-x}} = \frac{1}{\sqrt{4md}}\times\overline{a-x}|^{-\frac{1}{2}}\times -\dot{x}$, whoſe fluent (Art. 39.) is $t = \frac{2}{\sqrt{4md}}\times\overline{a-x}|^{\frac{1}{2}} + C$; but when $t=0$, $x=a$, $\therefore$ $C=0$; hence $t = \frac{2}{\sqrt{4md}}\times\overline{a-x}|^{\frac{1}{2}}$.

Ex. 2. If $n=1$, then $v = \sqrt{2md}\times\sqrt{a^2-x^2} =$



$\sqrt{2md}\times CD$, if upon SA a quadrant be deſcribed, and the ordinate CD be erected $\perp$ to AS. Alſo, $\dot{t} = \sqrt{\frac{1}{2md}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}}$; but if $z = AD$, then (Art. 46.) $\dot{z} : -\dot{x} :: a : \sqrt{a^2 - x^2}$, $\therefore \frac{-\dot{x}}{\sqrt{a^2 - x^2}} = \frac{\dot{z}}{a}$; hence $\dot{t} = \sqrt{\frac{1}{2md}} \times \frac{\dot{z}}{a}$, whoſe fluent (which wants no correction, becauſe when $t = 0$, $z = 0$) is $t = \sqrt{\frac{1}{2md}} \times \frac{z}{a}$, the time through AC; hence if $p = 1{,}57079$ (which is $\frac{1}{4}$ of the circumference of a circle whoſe radius $= 1$), we have $\sqrt{\frac{1}{2md}} \times p$ for the whole time through AS, becauſe z then $= AV = pa$. Hence, from whatever diſtance the body falls, the whole time of deſcent will be the ſame, it being independent of AS.

Ex. 3. If $n = -2$, $v = \sqrt{4md} \times \sqrt{x^{-1} - a^{-1}} = \sqrt{4md} \times \sqrt{\frac{a-x}{ax}}$. Alſo, $\dot{t} = -\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{-x\dot{x}}{\sqrt{ax - x^2}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \overline{\frac{\frac{1}{2}a\dot{x} - x\dot{x}}{\sqrt{ax - x^2}} - \frac{\frac{1}{2}a\dot{x}}{\sqrt{ax - x^2}}}$, whoſe fluent (Art. 40. and 46.) is $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ $\overline{\sqrt{ax - x^2}}$ − a cir. arc whoſe rad. $= \frac{1}{2}a$ and verſed ſine x $+ C =$ (if upon AS we deſcribe a ſemicircle) $\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \overline{CE - SE} + C$; but when $t = 0$, this becomes $0 =$

$0 = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times -\text{arc } SEA + C, \therefore C = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \text{arc } SEA;$

conſequently $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \overline{CE + \text{arc } AE}$. Hence the whole time to $S = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \text{ arc } AES$.

Ex. 4. If $n = -3$, $v = \sqrt{2md} \times \sqrt{x^{-2} - a^{-2}} = \sqrt{2md} \times \frac{\sqrt{a^2 - x^2}}{ax}$. Alſo, $\dot{t} = \frac{1}{\sqrt{2md}} \times \frac{-ax\dot{x}}{\sqrt{a^2 - x^2}}$; and therefore $t = \frac{1}{\sqrt{2md}} \times a\sqrt{a^2 - x^2} = \frac{1}{\sqrt{2md}} \times AS \times CD$, which wants no correction, becauſe when $t = 0$, $CD = 0$, and both ſides vaniſh together. Hence the whole time of deſcent to $S = \frac{1}{\sqrt{2md}} \times AS^2$.

Ex. 5. If $e = 1$, $c = r$ the radius of the Earth, $n = -2$, and a be taken any diſtance from the Earth's center greater than r, then $d = r^2$, and $v = \sqrt{4mr^2} \times \sqrt{\frac{a-x}{ax}} = r\sqrt{4m} \times \sqrt{\frac{a-x}{ax}}$ the velocity acquired in falling from any diſtance a from the center through $a - x$; and when $x = r$, $v = r\sqrt{4m} \times \sqrt{\frac{a-r}{ar}} = \sqrt{4mr} \times \sqrt{\frac{a-r}{a}}$ the velocity acquired in falling through the ſpace $a - r$ to the Earth's ſurface. If a be infinite, $v = \sqrt{4mr}$ the velocity which a body would acquire in falling from an infinite diſtance.

Ex. 6. If $n = 1$, and $a = r$; then $d = \frac{1}{r}$; hence $v = \sqrt{\frac{2m}{r} \times \overline{r^2 - x^2}}$, and when $x = 0$, $v = \sqrt{2mr}$, which is the velocity a body acquires in falling from the ſurface of the Earth to the center, becauſe within the Earth's ſurface the force varies directly as the diſtance. Alſo, by Ex. 2. $t=p\times\sqrt{\frac{1}{2md}}=p\sqrt{\frac{r}{2m}}$. Hence, by SIR I. NEWTON's *Principia*, *Lib. I. p.* 38. *Cor.* 1. the time in which a body would revolve about the Earth at it's ſurface $=4p\sqrt{\frac{r}{2m}}=p\sqrt{\frac{8r}{m}}$. This is the time in ſeconds; alſo $4pr$ is the circumference of the Earth; hence $p\sqrt{\frac{8r}{m}} : 4pr :: 1'' : \sqrt{2rm}$ the velocity of a body revolving about the Earth in a circle at it's ſurface, the velocity being always meaſured by the ſpace deſcribed uniformly in 1″. We muſt take r in feet, m being in feet. Hence it appears, that the velocity of a body falling from the ſurface of the Earth to it's center is equal to the velocity of a body revolving at the Earth's ſurface.

Cor. 1. From hence we may find how far a body muſt fall above the Earth's ſurface to acquire the velocity in a circle at the ſurface, ſuppoſing $n=-2$; for then, by the two laſt examples, $\sqrt{4mr}\times\sqrt{\frac{a-r}{a}}=\sqrt{2mr}$; hence $a=2r$, and $a-r=r$ the ſpace fallen through.

Cor. 2. Let s be the ſpace a body muſt fall through by the conſtant force of gravity at the Earth's ſurface to acquire the velocity $\sqrt{2rm}$ in a circle; then, by Mechanics, $v^2=4ms=2rm$, hence $s=\frac{1}{2}r$; and the ſame is true for any circle.

Ex. 7. If inſtead of ſuppoſing the body to fall from a ſtate of reſt at A, it be projected with a velocity b, then when $x=a$, $v=b$, therefore (Art. 73.) $\frac{b^2}{2} = -\frac{2md}{n+1} \times a^{n+1} + C$; hence $C = \frac{b^2}{2} + \frac{2md}{n+1} \times a^{n+1}$; conſequently $v = \sqrt{b^2 + \frac{4md}{n+1} \times \overline{a^{n+1} - x^{n+1}}}$. Now to find to what height the body will aſcend if it be projected upwards, we muſt put $v=0$, and then $b^2 + \frac{4md}{n+1} \times \overline{a^{n+1} - x^{n+1}} = 0$; hence $x = \overline{\frac{n+1}{4mb} \times b^2 + a^{n+1}}\Big|^{\frac{1}{n+1}}$, the diſtance from the center of force to which the body aſcends.

Ex. 8. If $n = -1$, the fluent thus taken fails (Art. 38.) and $v\dot{v} = -2md \times \frac{\dot{x}}{x}$, whoſe fluent (Art. 45.) is $\frac{v^2}{2} = -2md \times \text{h. l. } x + C$; but when $v=0$, $x=a$, and the equation becomes $0 = -2md \times \text{h. l. } a + C$, hence $C = 2md \times \text{h. l. } a$; therefore $v = \sqrt{4md} \times \sqrt{\text{h. l. } a - \text{h. l. } x} = \sqrt{4md} \times \sqrt{\text{h. l. } \frac{a}{x}}$.

ON THE MOTION OF BODIES IN RESISTING MEDIUMS.

74. If a cylinder move in a fluid in the direction of it's axis with the velocity d, the reſiſtance is equal to the weight of a column of fluid whoſe baſe is equal to the end of the cylinder, and altitude k through which a body muſt fall to acquire the velocity d. This appears from experiment; it has been attempted to be proved independent of experiment, but the proofs are by no means ſatisfactory.

75. The reſiſtance of a globe of the ſame diameter as the cylinder and moving with the ſame velocity, is half the reſiſtance of the cylinder. Now let $p=$ 0,78539 &c. $h=$ the diameter of the globe, $m=$ $16\frac{1}{12}$ feet, and put the denſity of the globe : the denſity of the medium :: $n : 1$; then the magnitude of the globe : magnitude of a column of fluid equivalent to the reſiſtance of the globe :: $\frac{2}{3}ph^3 : \frac{1}{2}ph^2k ::$ (becauſe, by Mechanics, $k=\frac{d^2}{4m}$) $h : \frac{3d^2}{16m}$; hence their quantities of matter are as $nh : \frac{3d^2}{16m}$, or $1 : \frac{3d^2}{16mnh}$; therefore if the weight of the globe, or it's gravity, be denoted by unity, $\frac{3d^2}{16mnh}$ will repreſent it's reſiſtance moving with the velocity d. Hence the reſiſtance of the cylinder is $\frac{3d^2}{8mnh}$.

PROP. XXXVI.

Let a globe be projected in a reſiſting medium, as in the laſt article, and let the reſiſtance be as the c^{th} power of the the velocity; to find the velocity v, time t, and ſpace x deſcribed, any one in terms of the other.

76. Let d be the velocity of projection, and r the reſiſtance correſponding to the velocity d compared with the force of gravity repreſented by unity; then $r=\frac{3d^2}{16mnh}$ by the laſt article. Hence $d^c : v^c :: r : \frac{r}{d^c}\times v^c$ the reſiſtance correſponding to the velocity v; therefore (Art. 72.) $v\dot{v}=-\frac{2mr}{d^c}\times v^c\dot{x}=\left(\text{if } \frac{1}{e}=\frac{2mr}{d^c}\right)$ $-\frac{1}{e}v^c\dot{x}$, hence $\dot{x}=-ev^{1-c}\dot{v}$, conſequently $x=-\frac{e}{2-c}\times v^{2-c}+C$; but when $x=0$, $v=d$, and the equation becomes $0=-\frac{e}{2-c}\times d^{2-c}+C$, hence $C=\frac{e}{2-c}\times d^{2-c}$; therefore $x=\frac{e}{2-c}\times\overline{d^{2-c}-v^{2-c}}$. Hence, when $v=0$, and c is leſs than 2, $x=\frac{e}{2-c}\times d^{2-c}$, the the whole ſpace deſcribed before the velocity is all deſtroyed.

If $c=2$, the fluent fails; for then $\dot{x}=-\frac{e\dot{v}}{v}$, and $x=e\times\text{h. l. }\frac{d}{v}=\left(\text{becauſe } e=\frac{8nh}{3}\right)\frac{8nh}{3}\times\text{h. l. }\frac{d}{v}$. Hence when $v=0$, x becomes infinite, therefore the velocity will never be deſtroyed.

If c be greater than 2, $2-c$ is negative, and by making $v=0$, x becomes a negative quantity, which ſhows that the velocity will never be all deſtroyed.

Alſo (Art. 72.) $\dot{t} = \frac{\dot{x}}{v} = -ev^{-c}\dot{v}$, and $t = -\frac{e}{1-c} \times v^{1-c} + C$; but when $t=0$, $v=d$, hence $C = \frac{e}{1-c} \times d^{1-c}$; therefore $t = \frac{e}{1-c} \times \overline{d^{1-c} - v^{1-c}}$. Hence when $v=0$, and c is leſs than 1, $t = \frac{e}{1-c} \times d^{1-c}$, the time of deſcribing the whole ſpace.

If $c=1$, the fluent fails, for then $\dot{t} = \frac{-e\dot{v}}{v}$, whoſe fluent corrected is $t = e \times \text{h. l.}\ \frac{d}{v}$. Hence when $v=0$, t becomes infinite. But it appears that, in this caſe, the ſpace is finite; hence the body is an infinite time in deſcribing a finite ſpace, and which ſpace is ed.

If c be greater than 1, then $1-c$ is negative, and when $v=0$, t becomes a negative quantity, which ſhows the time to be infinite; but the ſpace will ſtill be finite whilſt c is leſs than 2. When therefore c is equal to, or greater than 2, both the ſpace and time will be infinite.

As $v = \overline{d^{2-c} - \frac{2-c}{e} \times x}\Big|^{\frac{1}{2-c}}$, ſubſtitue this quantity for v, and it gives $t = \frac{e}{1-c} \times$

$d^{1-c} -$

$\overline{d^{1-c}-\overline{d^{2-c}-\frac{2-c}{e}\times x}\Big|^{\frac{1-c}{2-c}}}$, ſhowing the relation between t and x, except in the caſes where the fluents fail.

Prop. XXXVII.

Let a body be projected in a reſiſting medium directly to or from a center of force, and be attracted by a conſtant force towards that center; to find the ſpace, time, and velocity.

77. Let F be the force compared with gravity which is repreſented by unity, and retain the notation in Art. 76. Now when the body *deſcends*, the whole accelerative force $= F -$ the reſiſtance, and when it *aſcends*, the retarding force $= F +$ the reſiſtance; that is, in the *former* caſe the force $= F - \frac{r}{d^c}\times v^c$, and in the *latter* it $= F + \frac{r}{d^c}\times v^c$. Hence (Art. 72.) $v\dot{v} = \pm 2m \times \overline{F \mp \frac{r}{d^c}\times v^c} \times \dot{x}$, the *upper* ſigns being uſed when the body *deſcends*, and the *lower* when it *aſcends*; hence (if $\frac{r}{d^c} = e$) $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^c}$.

If $c = 2$, $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^2}$, whoſe fluent (Art. 45.) is $x = \frac{1}{2m} \times \frac{1}{2e} \times -$ h. l. $\overline{F \mp ev^2} + C$; but when $x = 0$, $v = d$, and the fluent becomes $0 = \frac{1}{4me} \times -$

h. l. $\overline{F\mp ed^2}+C$, hence $C=\frac{1}{4me}\times$ h. l. $\overline{F\mp ed^2}$; conſequently $x=\frac{1}{4me}\times$ h. l. $\frac{F\mp ed^2}{F\mp ev^2}$. Hence we may find v in terms of x; for $4mex=$ h. l. $\frac{F\mp ed^2}{F\mp ev^2}$; therefore put $w=$ the number whoſe h. l. is $4mex$, and then $w=\frac{F\mp ed^2}{F\mp ev^2}$, hence $v=\sqrt{\frac{F\mp ed^2-wF}{\mp we}}$.

78. If the body *aſcend*, and $v=0$, $x=\frac{1}{4me}\times$ h. l. $\frac{F+ed^2}{F}$ the diſtance to which it aſcends.

79. Let the body *deſcend*. Now when $F=\frac{rv^2}{d^2}$, the reſiſtance becomes equal to the accelerating force; hence $v^2=\frac{F\times d^2}{r}$, and $v=d\sqrt{\frac{F}{r}}$, the greateſt velocity the body can acquire, for when the reſiſtance becomes equal to the attractive force, there can be no further acceleration.

80. If $d=0$, $x=\frac{1}{4me}\times$ h. l. $\frac{F}{F\mp ev^2}$.

81. Alſo (Art. 72.) $\dot{t}=\frac{\dot{x}}{v}=\frac{1}{2m}\times\frac{\pm\dot{v}}{F\mp ev^2}$; hence when the body *deſcends*, $\dot{t}=\frac{1}{2me}\times\frac{\dot{v}}{\frac{F}{e}-v^2}$, whoſe fluent (Art. 45.), (putting $\frac{F}{e}=a^2$) is $t=\frac{1}{4mae}\times$ h. l. $\frac{a+v}{a-v}+C$; but when $t=0$, $v=d$, and we get $0=$

$\frac{1}{4mae}\times$ h. l. $\frac{a+d}{a-d}+C$, hence $C=-\frac{1}{4mae}\times$ h. l. $\frac{a+d}{a-d}$;

consequently $t=\frac{1}{4mae}\times\overline{\text{h. l. }\frac{a+v}{a-v}-\text{h. l. }\frac{a+d}{a-d}}$; Hence if we ſubſtitute the value of v in terms of x, we ſhall get t in terms of x. If the body fall from a ſtate of reſt, $t=\frac{1}{4mae}\times$ h.l. $\frac{a+v}{a-v}$.

82. When the body *aſcends*, $\dot{t}=\frac{1}{2m}\times\frac{-\dot{v}}{F+ev^2}=\frac{1}{2me}\times\frac{-\dot{v}}{a^2+v^2}$, whoſe fluent (Art. 46.) is $t=\frac{1}{2mea}\times -M+C$, M being a circular arc whoſe radius is 1, and tangent $\frac{v}{a}$; but when $t=0$, $v=d$; put therefore $N=$ the arc whoſe tangent is $\frac{d}{a}$, and we get $t=\frac{1}{2mea}\times\overline{N-M}$. For the whole aſcent, $v=0$, $\therefore M=0$; hence $t=\frac{1}{2mea}\times N$.

83. If we apply theſe expreſſions to the deſcent of bodies in reſiſting mediums upon the Earth's ſurface, then as unity repreſents our force of gravity, that is, the force when a body falls in vacuo, we muſt find the value of F when a body deſcends in the medium. Let the denſity of the body : the denſity of the medium :: n : 1; then if $w=$ the weight of the body in vacuo, we have, by Hydroſtatics, w : weight loſt when in the fluid :: n : 1, hence w : $w-$ weight loſt, or weight in the fluid, :: n : $n-1$, therefore the weight

in the fluid $= w \times \frac{n-1}{n} =$ (if $w = 1$ the force of gravity) $\frac{n-1}{n}$ which is the gravity of the body in the fluid, or the force with which it endeavours to deſcend; this therefore is the value of *F*.

84. By Art. 76. $r = \frac{3d^2}{16mnh}$; hence $a\left(= \sqrt{\frac{F}{e}} = d\sqrt{\frac{F}{r}}\right) = \sqrt{\frac{16.\overline{n-1}.mh}{3}}$, which is the greateſt velocity the body can acquire by falling in the fluid, for (Art. 79.) that velocity is $= d\sqrt{\frac{F}{r}}$. Alſo $t = \frac{1}{4mae} \times \text{h. l.}\, \frac{a+v}{a-v}$; and when $v = a$, t becomes infinite; therefore the body never can acquire it's greateſt velocity.

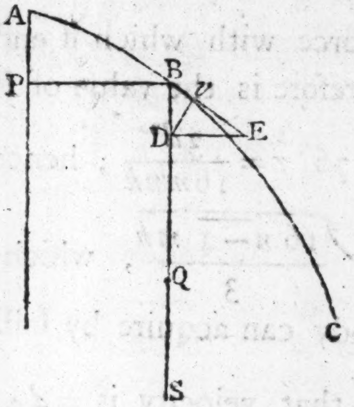
85. The greateſt height to which a body can aſcend when projected upwards is (Art. 78.) $\frac{1}{4me} \times \text{h. l.}\, \overline{\frac{F + ed^2}{F}} = \frac{4nh}{3} \times \text{h. l.}\, \overline{1 + \frac{3d^2}{16.\overline{n-1}.mh}}$.

Prop. XXXVIII.

To determine the reſiſtance of a medium, by which a body may deſcribe any curve about a center of force, the force to the center being given.

86. Let *ABC* be the given curve, *S* the center of force, and *F* the force of the body at *B* towards it,

the force of gravity being unity; draw DE perpendicular to BS, meeting the tangent BE; and Dv perpendicular to BE. Put $AB=z$, $BS=w$, $BD=-\dot{w}$,



$BE=\dot{z}$, $v=$ the velocity in the curve at B, and $s=BQ=\frac{1}{2}$ the chord of the circle of curvature at B paſſing through S, $m=16\frac{1}{12}$ feet. Now it is well known that a body, whether it moves in a reſiſting medium or not, muſt fall down $\frac{1}{2}s$ by the conſtant force F to acquire the velocity in the curve; for the reſiſtance cauſes no deviation from the tangent, but only retards the motion of the body, ſo that it may preſerve it's proper proportion correſponding to the force; hence, by Mechanics, $v^2=4mF\times\frac{1}{2}s=2mFs$; therefore $\dot{v}=m\times\frac{F\dot{s}+s\dot{F}}{\sqrt{2mFs}}$ the *whole* increaſe of velocity in the direction BE. But by Mechanics, the velocity v which the force F continuing conſtant for any time t, would generate in the direction BS is $2mFt$; $\therefore \dot{v} = 2mF\dot{t} =$ (becauſe $\dot{t} = \frac{BE}{v} = \frac{\dot{z}}{v}$) $m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$ the increaſe of the velocity in the direction BS ariſing from the force F; hence $BD : Bv$ ($:: BE = \dot{z} : BD = -\dot{w}$) $:: m \times \frac{2F\dot{z}}{\sqrt{2mFs}} : m \times \frac{-2F\dot{w}}{\sqrt{2mFs}}$ the increaſe of velocity in the direction BE ariſing from the force F; from which if we take the whole increaſe $m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}}$ there will remain $-m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}}$, which ſhows the decrement of velocity ariſing from the reſiſtance, in the time that the force F would generate the velocity $m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$; but the velocity generated or deſtroyed in the ſame time muſt be in proportion to the force; hence the reſiſtance : force F $:: m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}} : m \times \frac{2mF\dot{z}}{\sqrt{2mFs}} :: \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{2F\dot{z}} : 1$, omiting the ſign $-$ before the firſt term, as it only ſignifies the force to be retarding.

87. When the center S is at an infinite diſtance, and the force F becomes conſtant and acts in parallel lines; then $\dot{F} = 0$, and the reſiſtance : force $F :: \frac{\dot{s} + 2\dot{w}}{2\dot{z}} : 1$. But if we draw AP parallel to BS, and PB perpendicular to it, and put $AP = x$, then $\dot{w} = -\dot{x}$; hence the reſiſtance : force $F :: \frac{\dot{s} - 2\dot{x}}{2\dot{z}} : 1$. Or to obtain this proportion in terms of the abſciſſa and curve, put $y = PB$; then by Art. 53.

$\dot{z}^2=\dot{x}^2+\dot{y}^2$; and by Art. 97. $s=\frac{\dot{z}^2}{\ddot{x}}=\frac{\dot{x}^2+\dot{y}^2}{\ddot{x}}$, therefore if we ſuppoſe $\dot{y}$ conſtant, we ſhall have $\dot{s}=\frac{2\dot{x}\ddot{x}^2-\overline{\dot{x}^2+\dot{y}^2}\times\dddot{x}}{\ddot{x}^2}=\frac{2\dot{x}\ddot{x}^2-\dot{z}^2\dddot{x}}{\ddot{x}^2}$; hence $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=-\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$; therefore the reſiſtance : force F :: $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$: 1.

Examples.

Ex. 1. Let the curve be a *parabola*, and the force be conſtant and act in lines parallel to AP. Then if $x=AP$, $y=PB$, $ax=y^n$, ∴ $a\dot{x}=ny^{n-1}\dot{y}$, and ($\dot{y}$ being conſtant) $a\ddot{x}=n.\overline{n-1}.y^{n-2}\dot{y}^2$, alſo $\dddot{x}=\frac{n.\overline{n-1}.\overline{n-2}}{a}\times y^{n-3}\dot{y}^3$, and $\dot{z}=\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$; hence $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}=\frac{n-2}{2.n.\overline{n-1}}\times\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}}{y^{n-1}}$, the reſiſtance.

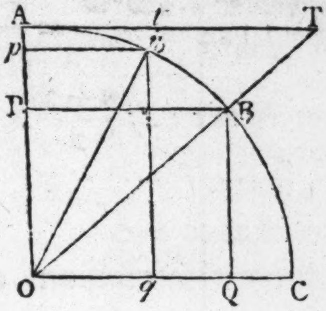
If $n=2$, the reſiſtance becomes $=0$.

If n be leſs than 2, but greater than 1, the reſiſtance becomes negative, the medium therefore muſt propel the body, not retard it.

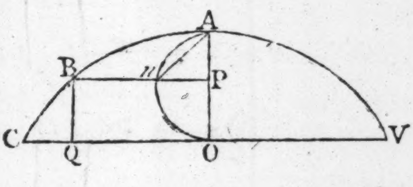
If $n=1$, the medium becomes an infinite propelling one, and the body moves in a right line.

Ex. 2. Let ABC be a *quadrant of a circle.* Put $AO=a$, $AP=x$, $AB=z$, then $BQ=s=a-x$, and $\dot{s}=-\dot{x}$; hence $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=-\frac{3\dot{x}}{2\dot{z}}=\frac{3PB}{2OB}=$ the reſiſtance, gravity being unity. Hence at A the reſiſtance $=0$; when $3PB=2BO$, or radius : ſine of AB :: 3 : 2, the reſiſtance = gravity; and at C, the reſiſtance :

gravity :: 3 : 2. Alſo the velocity is as $\sqrt{BQ}$. Hence alſo the reſiſtance at $B \propto PB$. Now if we ſuppoſe the reſiſtance to vary as the denſity of the medium × the ſquare of the velocity, then the denſity varies as the reſiſtance directly and ſquare of the velocity inverſely, or as $\frac{PB}{BQ} = \frac{PB}{PO} = \frac{AT}{AO}$; hence the denſity at B varies as the tangent of AB. All this agrees with what SIR I. NEWTON has proved in his *Principia Lib.* 2. *Sec.* 2. *Pr.* 10.



Ex. 3. Let *CAV* be a *cycloid*. Then $BQ = \frac{1}{2}s$, and if $AO = a$, $\frac{1}{2}s = a - x$, $\therefore \frac{1}{2}\dot{s} = -\dot{x}$, and $\dot{s} = -2\dot{x}$, alſo $\dot{z} = \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}}$ (Art. 53. Ex. 2); hence $\frac{\dot{s} - 2\dot{x}}{2\dot{z}} = \frac{-2x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ = (becauſe $x : An :: An : AO = a$) $\frac{2An}{AO}$ the reſiſtance, gravity being unity. Alſo the velocity varies as $\sqrt{BQ}$.



Ex. 4.

Ex. 4. Let the force tend to a center S, and vary as w^n, and the curve to be the *logarithmic ſpiral*. Then as $F = w^n$, $\dot{F} = n w^{n-1} \dot{w}$; alſo $s = w$, $\therefore \dot{s} = \dot{w}$; hence the reſiſtance $= \frac{w^n \dot{w} + n w^n \dot{w} + 2 w^n \dot{w}}{2 w^n \dot{z}} = \frac{n+3}{2} \times \frac{\dot{w}}{\dot{z}} =$ (as w : $\dot{z}$ in ſome conſtant ratio $c : d$) $\frac{n+3}{2} \times \frac{c}{d}$, the force tending to S being unity.

If $n = -3$ the reſiſtance $= 0$.

If $n+3$ be negative the medium muſt propel the body. Alſo $v = \sqrt{2mFs} = \sqrt{2m} \times w^{\frac{n+1}{2}}$. Now the reſiſtance being to the force F, as $\frac{n+3}{2} \times \frac{c}{d}$ to 1, if F be repreſented by it's true value w^n, the reſiſtance will become $\frac{n+3}{2} \times \frac{c}{d} \times w^n$; and ſince the denſity of the medium varies as the reſiſtance directly and the ſquare of the velocity inverſely, the denſity varies as $\frac{w^n}{w^{n+1}}$, or as $\frac{1}{w}$. Hence if the denſity of the medium vary inverſely as the diſtance, the body may deſcribe the logarithmic ſpiral by any force, agreeable to what SIR I. NEWTON has proved in his *Principia*, *Lib.* 2. *Sec.* 4. *Prop.* 16. If $n = -2$, $F = \frac{1}{w^2}$, or F varies as the ſquare of the denſity, as he has alſo proved in *Prop.* 15.

SECTION VIII.

ON SECOND, THIRD, &c. FLUXIONS.

Prop. XXXIX.

To explain under what circumſtances a quantity may have ſeveral orders of fluxions.

88. The fluxion of a quantity being the uniform increaſe or decreaſe of that quantity in a given time, every quantity which increaſes or decreaſes muſt have a fluxion. Hence if the fluxion of any quantity be not conſtant, it muſt have ſome certain rate of increaſe or decreaſe, which rate of increaſe or decreaſe will therefore be the fluxion of that fluxion, or the ſecond fluxion of the original flowing quantity. Alſo if this ſecond fluxion be not always the ſame, it muſt have a rate of variation, that rate therefore will be the fluxion of the ſecond fluxion, or the third fluxion of the original quantity; and ſo on. Thus a quantity will have a ſucceſſive order of fluxions till

ſome one fluxion becomes conſtant, and then by Art. 3. it will have no more. Thus, let x increaſe uniformly; then the fluxion of x^2 is $2x\dot{x}$; now $\dot{x}$ is conſtant, but x itſelf increaſes, therefore $2x\dot{x}$ increaſes in proportion to the increaſe of x; the fluxion therefore of x^2 is not conſtant. Hence, conſidering x as the variable part of $2x\dot{x}$, it's fluxion by Art. 9. is $2\dot{x}\dot{x} = 2\dot{x}^2$, which is the ſecond fluxion of x^2. But if we ſuppoſe x not to increaſe uniformly, then $2x\dot{x}$ will have both x and $\dot{x}$ variable; hence by Art. 14. the fluxion of $2x\dot{x}$ will be $2\dot{x}\dot{x}+2x\ddot{x}$, or $2\dot{x}^2+2x\ddot{x}$, which therefore is the ſecond fluxion of x^2, $\ddot{x}$ denoting the fluxion of $\dot{x}$, or the ſecond fluxion of x. In like manner we denote the third, fourth, &c. fluxions of x by $\dddot{x}$; $\ddddot{x}$, &c. But if we ſhould here ſuppoſe neither $\dot{x}$ nor $\ddot{x}$ to be conſtant, then this ſecond fluxion would be variable. Now the fluxion of $2\dot{x}^2$ is found by Art. 11. conſidering here $\dot{x}$ as the root, and therefore the fluxion of the root is $\ddot{x}$; hence the fluxion of $2\dot{x}^2$ is $4\dot{x}\ddot{x}$; alſo the fluxion of $2x\ddot{x}$ is found by Art. 9. both x and $\ddot{x}$ being variable; hence it's fluxion is $2\dot{x}\ddot{x}+2x\dddot{x}$; therefore the fluxion of $2\dot{x}^2+2x\ddot{x}$, or the third fluxion of x^2, is $4\dot{x}\ddot{x}+2\dot{x}\ddot{x}+2x\dddot{x} = 6\dot{x}\ddot{x}+2x\dddot{x}$. In like manner we may find the ſucceſſive orders of fluxions of any quantity.

89. If x increaſe uniformly, or if $\dot{x}$ be conſtant, x^n will have n fluxions, and no more, n being an affirmative whole number. For the firſt fluxion is $nx^{n-1}\dot{x}$, and x only being variable, it's fluxion is $n.\overline{n-1}.x^{n-2}\dot{x}^2$; and the fluxion of this is $n.\overline{n-1}.\overline{n-2}.x^{n-3}\dot{x}^3$ &c. when therefore we have taken the fluxion n times, the

index of x becomes $=0$, and $x^0=1$; hence the fluxion then becomes $n.\overline{n-1}\ldots 2.1.\dot{x}^n$, which being a conſtant quantity, it has no further fluxion.

90. The firſt fluxion of x^3+ay^2 is $3x^2\dot{x}+2ay\dot{y}$, and if $\dot{x}$ and $\dot{y}$ be both variable, it's fluxion is $6x\dot{x}^2+3x^2\ddot{x}+2a\dot{y}^2+2ay\ddot{y}$; but if $\dot{x}$ be conflant, then $\ddot{x}=0$; therefore the ſecond fluxion becomes $6x\dot{x}^2+2a\dot{y}^2+2ay\ddot{y}$; and if $\dot{y}$ be conſtant, the ſecond fluxion is $6x\dot{x}^2+3x^2\ddot{x}+2a\dot{y}^2$.

91. The firſt fluxion of $x^n y^m$, by Art. 9. is $ny^m x^{n-1}\dot{x}+mx^n y^{m-1}\dot{y}$; and if both $\dot{x}$ and $\dot{y}$ be variable, we are to conſider each of theſe quantities as compoſed of three variable factors, and then the fluxion, by the ſame Art. will be $n.mx^{n-1}y^{m-1}\dot{y}\dot{x}+n.\overline{n-1}.y^m x^{n-2}\dot{x}^2+ny^m x^{n-1}\ddot{x}+m.\overline{m-1}.x^n y^{m-2}\dot{y}^2+mny^{m-1}x^{n-1}\dot{x}\dot{y}+mx^n y^{m-1}\ddot{y}$.

ON THE POINT OF CONTRARY FLEXURE OF A CURVE.

DEFINITION.

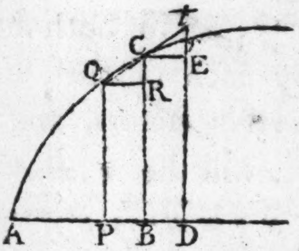
92. If a curve be concave on one part and convex on another, the point where the concave part ends and the convex begins, is the point of contrary flexure.

PROP.

Prop. XL.

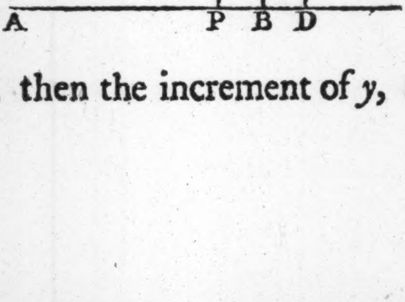
To find the point of contrary flexure of a curve.

93. Let PQ, BC, Dr, be three equidiſtant ordinates, and the curve concave to the axis; and draw QR, CE parallel to AD, and join QC, and produce it to meet Dr in t. Then the triangles QRC, CEt being ſimilar, and $QR = CE$, therefore $CR = tE$, and



hence CR is greater than Er; therefore if y repreſent the ordinate, moving from A, and x the abſciſſa, and $PB = BD = \dot{x}$ a conſtant quantity; then correſponding to the uniform increaſe of x, the increment of y, and conſequently $\dot{y}$ decreaſes; now as y increaſes, $\dot{y}$ is poſitive by Art. 16. but as $\dot{y}$ decreaſes, it's fluxion, or $\ddot{y}$, is negative by the ſame article.

If the curve be convex to the axis, and the ordinate



move from A, then the increment of y, and therefore

$\dot{y}$ increaſes; and as y increaſes, $\dot{y}$ is poſitive, and, as $\dot{y}$ increaſes, it's fluxion, or $\ddot{y}$ is poſitive. Therefore when the curve is *concave* to the axis, $\ddot{y}$ is *negative*; if *convex*, $\ddot{y}$ is *poſitive*, $\dot{x}$ being conſtant. Hence at the point of contrary flexure $\ddot{y}$ changes it's ſign; but a quantity may change it's ſign either by paſſing through 0, or *infinity*; hence at the point of contrary flexure $\ddot{y}=0$, or *infinity*. What we here mean by infinity is only in reſpect to it's value at any other time, that term being relative; and in this caſe we are to underſtand that $\ddot{y}$ is indefinitely greater at that time than at any other. If we conceive a line to be drawn from A parallel to BC, and conſider that as an abſciſſa to the curve, and draw lines from it to Q, C, r parallel to AD; then the former abſciſſæ AP, AB, AD become equal to the ordinates, and the ordinates PQ, BC, Dr become equal to the abſciſſæ; if therefore $\dot{y}$ be made conſtant, $\ddot{x}=0$, or *infinity* at the point of contrary flexure. Hence we have the following

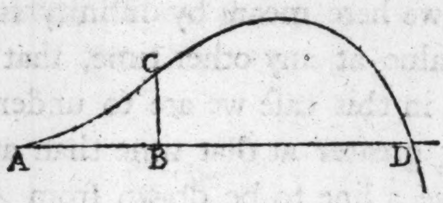
RULE:

Put the equation of the curve into fluxions, make $\dot{x}$ or $\dot{y}$ conſtant and take the fluxion of the equation again, and get the value of $\ddot{y}$ or $\ddot{x}$, and put it $=0$, or infinity; from which find the value of x, which gives the abſciſſa correſponding to the point of contrary flexure. And to determine for any value of x, whether the curve be concave or convex, ſubſtitute that value for x into the expreſſion for $\ddot{y}$, and if it come out poſitive, *the curve is* convex *to the axis*; *if* negative, *it is* concave.

Examples.

Examples.

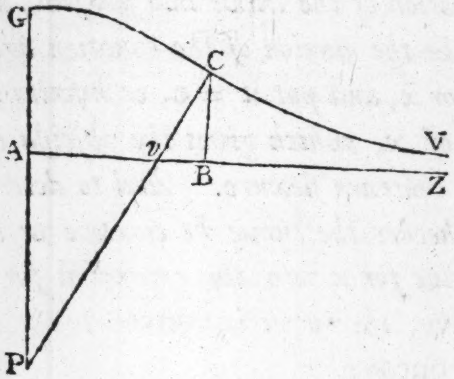
Ex. 1. Let the equation of the curve be $y=3x+18x^2-2x^3$; then $\dot{y}=3\dot{x}+36x\dot{x}-6x^2\dot{x}$, and $\ddot{y}=36\dot{x}^2-12x\dot{x}^2=$ (if $\dot{x}=1$) $36-12x$. Now make $36-12x=0$, and $x=3$; take therefore $AB=3$, and draw the ordinate BC, and C is the point of contrary flexure.



If x be between 0 and 3, $36-12x$ is poſitive, therefore the part AC of the curve is convex to AB; but when x is greater than 3, $36-12x$ is negative, and therefore the curve is concave towards the axis.

Ex. 2. Let GCV be a curve of ſuch a nature that if GA (which is perpendicular to AB) be produced to any point P, and PC be drawn to any point of the curve, vC ſhall always be equal to AG.

Put $AB=x$, $BC=y$, $PA=a$, $AG=b$; then by



ſim. trian. PAv, BCv, a (PA) : $x-\sqrt{b^2-y^2}$ $(AB-Bv)$

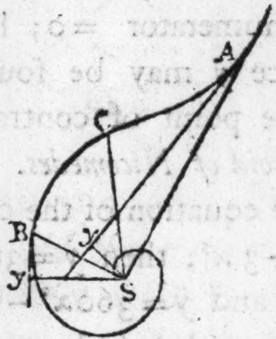
$Bv) :: y\ (BC) : \sqrt{b^2-y^2}\ (Bv)$; hence $xy = \overline{a+y} \times \sqrt{b^2-y^2}$; take the fluxion, and $y\dot{x} + x\dot{y} = \dot{y}\sqrt{b^2-y^2} - \frac{ay\dot{y} + y^2\dot{y}}{\sqrt{b^2-y^2}}$; ſubſtitute for x it's value, and we get $\dot{x} = -\frac{y^3+b^2a}{y^2\sqrt{b^2-y^2}} \times \dot{y}$; now make $\dot{y}$ conſtant, and we have $\ddot{x} = \frac{2b^4a - b^2y^3 - 3b^2ay^2}{\overline{b^2y^3-y^5} \times \sqrt{b^2-y^2}} \times \dot{y}^2$, which put $=0$, in which caſe the numerator $=0$; hence $y^3 + 3ay^2 = 2b^2a$; from whence y may be found, and then x, which will give the point of contrary flexure. This curve is the *Conchoid of Nicomedes*.

Ex. 3. Let the equation of the curve be $y = 180x^2 - 110x^3 + 30x^4 - 3x^5$; then $\dot{y} = 360x\dot{x} - 330x^2\dot{x} + 120x^3\dot{x} - 15x^4\dot{x}$, and $\ddot{y} = 360\dot{x}^2 - 660x\dot{x}^2 + 360x^2\dot{x}^2 - 60x^3\dot{x}^2 = 0$, or $-x^3 + 6x^2 - 11x + 6 = 0$, whoſe ſimple factors are $1-x$, $2-x$, $3-x$, whoſe roots are 1, 2, 3, the abſciſſæ correſponding to the points of contrary flexure, of which therefore there are three. As $-x^3 + 6x^3 - 11x + 6 = \overline{1-x} \times \overline{2-x} \times \overline{3-x}$, when x is leſs than 1, this quantity is poſitive, and therefore the curve is convex to the axis; when x is between 1 and 2, it is negative, and the curve is concave; when x is between 2 and 3, it is poſitive, and the curve is convex; when x is greater than 3 it is negative, and the curve will then continue concave.

94. If by making $\ddot{y} = 0$ the equation has 2 equal roots, then $\ddot{y}$ paſſes through 0 without changing it's ſign, in this caſe therefore the point found is not a point of contrary flexure. And this will always be the caſe when the equation has an *even* number of *equal* roots.

If the Reader wiſh to ſee any thing further upon this ſubject, he may conſult Mr. Lyons's *Fluxions*, page 136.

95. To find the point *C* of contrary flexure of a ſpiral, it is manifeſt, that as long as the point *A* approaches to *C*, the perpendicular *Sy* upon the tangent muſt increaſe; and after *A* has paſſed through *C* towards *S*, the perpendicular will then decreaſe; therefore at the point *C* it is a maximum; hence if we make the fluxion of the perpendicular $=0$, it will give the point of contrary flexure.

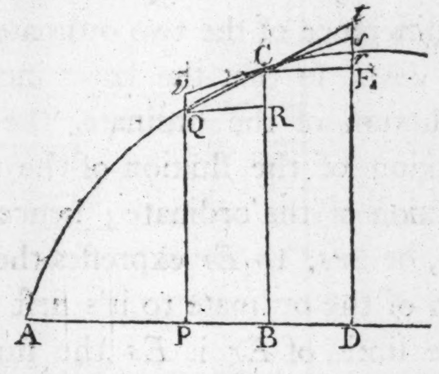


ON THE RADIUS OF CURVATURE.

PROP. XLI.

To find the ſecond fluxion of the ordinate of a curve.

96. Let PQ, BC, Dr be three equidiſtant ordinates, draw QR, CE parallel to AB, and let vCs be a tangent to C, meeting PQ, Dr in v and s; join QC and produce it to meet Ds in t. Now as $PB=BD$, the increment of the abſciſſa is conſtant, therefore (Art. 3. Cor. 1.) PB or BD will repreſent the fluxion of the abſciſſa, which is alſo conſtant. Now the cotemporary increments of the ordinates are RC, Er; but the triangles QRC, CEt are ſimilar, and $QR=CE$, therefore $RC=Et$, conſequently the cotemporary increments of the ordinates are Et, Er, and their difference rt; but as the limit of the increment or decrement of the ordinate is the fluxion of the ordinate (Art. 7.), therefore the limit of tr, the difference between two ſucceſſive increments of the ordinate, or the limit of the increment of the increment, will be the fluxion of the fluxion of the



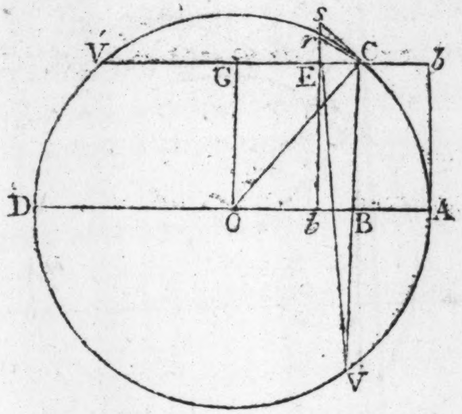
ordinate, or the ſecond fluxion of the ordinate. Now as the triangles CvQ, Cst are ſimilar, and $QC=Ct$, therefore $Qv=st$; and as Qv, sr depend upon the curvature of CQ, Cr, if Q and r be brought up to C, ſo as to get the meaſure of the curvature at C from each ſide, it is manifeſt that the *limit* of Qv to sr muſt be a ratio of equality; hence the *limiting* ratio of $rs : st$ is that of equality; conſequently the *limiting* ratio of $rt : 2rs$ is a ratio of equality. Hence if we take $2rs$ in two different parts of the curve and make them vaniſh, their limiting ratio expreſſes the ratio of the ſecond fluxions of the ordinates. Moreover, rt expreſſes the difference between the two ſucceſſive increments of the ordinates cotemporary with Er which expreſſes the difference of the two ordinates, therefore by taking the *limit*, ſo that the latter increment may become the fluxion of the ordinate, the former becomes the fluxion of the fluxion of the ordinate, or the ſecond fluxion of the ordinate; hence the *limit* of the ratio of rt, or $2rs$, to Er expreſſes the ratio of the ſecond fluxion of the ordinate to it's firſt fluxion; but (Art. 23.) the limit of Er is Es the fluxion of the ordinate, CE and Cs expreſſing the cotemporary fluxions of the abſciſſa and curve (Art. 27.); therefore the *limiting* ratio of $2rs : Cs$ and CE expreſſes the ratio of the ſecond fluxion of the ordinate : the fluxion of the curve AC and the fluxion of the abſciſſa AB. In like manner it appears if the curve be a ſpiral.

PROP.

PROP. XLII.

To find the radius of a circle in terms of the fluxions of it's abſciſſa, ordinate and curve.

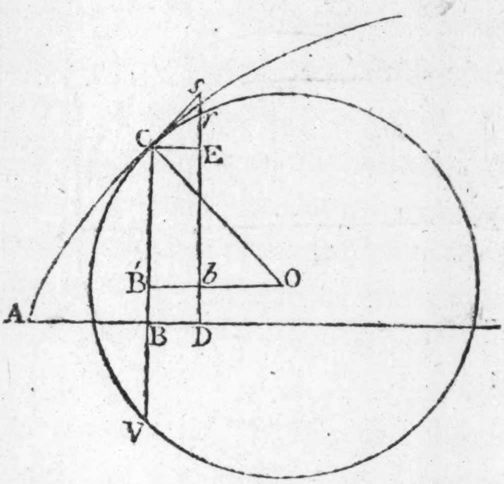
97. Let $ACDV$ be a circle, O the center, CBV perpendicular to AD, brs parallel to CB, Cs a tangent at C, and join rC, rV. Put $AB=x$, $BC=y$, $AC=z$, and $OC=a$. Now the triangles Crs, CVr are ſimilar, for the angle $srC=$ alter. ang. rCV, and the angle $sCr=$ angle CVr in the alternate ſegment; hence $sr:rC::rC:CV=2CB$; but by Art. 23. it appears that the *limiting* ratio of $rC:sC$ is a ratio of equality; therefore the *limiting* ratio of $sr:rC$ is $sr:sC$, or (Art. 96.) $-\frac{1}{2}\ddot{y}:\dot{z}$, the ſign $-$ being prefixedfor the reaſon in Art. 93. the curve being concave to the axis; hence $-\frac{1}{2}\ddot{y}:\dot{z}::\dot{z}:2BC$, $\therefore BC=\frac{\dot{z}^2}{-\ddot{y}}$; and by ſimilar triangles CEs, CBO, $\dot{x}:\dot{z}::\frac{\dot{z}^2}{-\ddot{y}}:CO=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, $\dot{x}$ being conſtant. If Ab be perpendicular to AO, and



bC to Ab; then conſidering Ab as the abſciſſa and bC the ordinate, we have, for the ſame reaſon, $CO = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, $\dot{y}$ being conſtant, and $\ddot{x}$ poſitive (Art. 93.), the curve being convex to the axis.

DEFINITION.

98. Let AC be any curve, AB the abſciſſa, BC the ordinate, Ds parallel to it, Cs a tangent at C, and let O be the center of a circle touching the curve in C, and draw OB' parallel to AB, and $DbErts$* parallel to BC; then if, by bringing Ds up to BC, the *limiting* ratio of $sr : st$ be a ratio of equality, the circle is ſaid to be a *circle of curvature to the curve*.



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* The letter t, denoting the interſection of Ds and the curve AC, is omitted in the Figure.

PROP.

Prop. XLIII.

To find the radius OC of the circle of curvature to the curve AC at the point C.

99. Whether we regard the curve *AC* or the circle, *CE*, *Es*, *Cs* will be the fluxions of the abſciſſa, ordinate and curve, for (Art. 23.) theſe fluxions depend entirely upon the poſition of the tangent, which is common to both; and by the Def. (Art. 98.) the *limiting* ratio of *sr* : *st* being a ratio of equality, the ſecond fluxions of the ordinates are equal (Art. 96.); hence the ſecond fluxion of the ordinate is the ſame whether we regard the curve or circle; but in the circle, if x, y and z repreſent the abſciſſa ordinate and curve, $CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, (Art. 97.) $\dot{x}$ being conſtant; hence in the curve, if x, y and z repreſent the abſciſſa *AB*, ordinate *BC* and curve *AC*, the radius $CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$. For the ſame reaſon $CO = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is conſtant.

When we make $\dot{x}$ or $\dot{y}$ conſtant, it will ſimplify the operation if we ſubſtitute unity for them.

Examples.

Ex. 1. Let *AC* be the common *parabola*; to find the radius of curvature.

Here $ax = y^2$, $\therefore y = a^{\frac{1}{2}} x^{\frac{1}{2}}$, and $\dot{y} = \frac{1}{2} a^{\frac{1}{2}} x^{-\frac{1}{2}}$, $\dot{x}$ being conſtant and $= 1$; hence $\ddot{y} = -\frac{1}{4} a^{\frac{1}{2}} x^{-\frac{3}{2}} = -\frac{a^{\frac{1}{2}}}{4x^{\frac{3}{2}}}$; alſo

$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{1 + \frac{a}{4x}} = \frac{1}{2}\sqrt{\frac{4x+a}{x}}$; therefore CO $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{4x+a}|^{\frac{3}{2}}}{2\sqrt{a}}$.

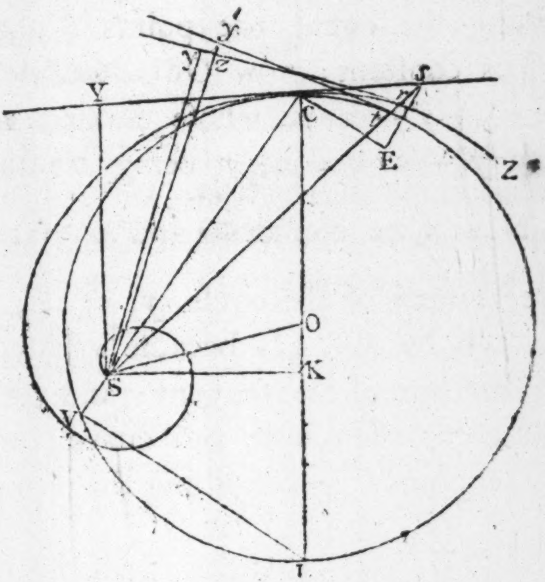
When $x = 0$, $CO = \frac{1}{2}a$, the radius of curvature at the vertex.

Ex. 2. Let it be the *logarithmic* curve; to find the radius of curvature.

By Art. 44. $\dot{y} = \frac{y\dot{x}}{m} =$ (if $\dot{x}$ be ſuppoſed conſtant and $= 1$) $\frac{y}{m}$, $\therefore \ddot{y} = \frac{\dot{y}}{m}$, and $-\dot{x}\ddot{y} = -\frac{\dot{y}}{m} = \frac{y}{m^2}$; alſo $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{1 + \frac{y^2}{m^2}} = \frac{\overline{m^2+y^2}|^{\frac{1}{2}}}{m}$; hence $CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{m^2+y^2}|^{\frac{3}{2}}}{-my}$, which being negative, ſhows that the center O lies on the other ſide of the curve, the curve being concave the other way.

TO FIND THE RADIUS OF CURVATURE TO SPIRALS.

100. Let CO be the radius of the circle of curvature to the ſpiral SCZ at C, and draw $Strs$ meeting the tangent YC in s; then by the Definition, Art. 98.) the *limiting* ratio of $sr : st$ is a ratio of equality, conſequently rt ultimately vaniſhes in reſpect to sr or st. Hence the tangents ry, ty' will ultimately form with each other an angle which becomes evaneſcent in reſpect to the angle formed by the tangents ry and sCY; therefore, ultimately, the difference zy' of the perpendiculars upon the tangents to r and t becomes evaneſcent in reſpect to the difference between SY and Sy; conſequently the *limit*



of the ratios of Sy and Sy' to $S\Upsilon$ muſt be the ſame; but the difference between $S\Upsilon$ and Sy', $S\Upsilon$ and Sy, or the increment of $S\Upsilon$ in each caſe, is ultimately the fluxion of $S\Upsilon$ in each caſe; hence the fluxion of the perpendicular to the curve and circle of curvature is the ſame.

Prop. XLIV.

To find the radius OC of the circle of curvature to the ſpiral at the point C.

101. Put $SC=y$, draw SK perpendicular to CO, and let $Sy=CK=v$, $CO=r$; now conſidering the point C as deſcribing the circle, the points S and O being fixed, SO is conſtant; now $OS^2=OC^2+CS^2-2OC \times CK = r^2+y^2-2rv$, whoſe fluxion therefore is $=0$, or $2y\dot{y}-2r\dot{v}=0$, r being conſtant; hence $r=\frac{y\dot{y}}{\dot{v}}$. Now if we conſider y and v in reference to the curve inſtead of the circle, $\dot{y}$, or sE, will be the ſame for each by Art. 31. becauſe sE depends only upon the poſition of the tangent; and (Art. 100.) $\dot{v}$ is the ſame for the circle and curve; hence if we conſider the point C as deſcribing the ſpiral, we ſhall ſtill have $r=\frac{y\dot{y}}{\dot{v}}$.

Examples.

Ex. 1. Let it be the *logarithmic* ſpiral; to find the radius of curvature.

Here $y : v :: m : n$, a conſtant ratio; hence $v=\frac{ny}{m}$, $\therefore \dot{v}=\frac{n\dot{y}}{m}$; hence $CO=y\dot{y}\times\frac{m}{n\dot{y}}=\frac{my}{n}$.

If we want to find the chord CV of the circle of curvature paſſing through S, we have $CL = \frac{2my}{n}$, and the triangles SCY, CLV being ſimilar, $y : v$, or $m : n$, $:: \frac{2my}{n} : CV = 2y = 2CS$.

Ex. 2. Let it be the ſpiral of *Archimedes*; to find the radius of curvature.

By Art. 32. $v = \frac{y^2}{\sqrt{y^2+b^2}}$; hence $\dot{v} = 2y\dot{y} \times \overline{y^2+b^2}^{-\frac{1}{2}}$ $- y\dot{y} \times \overline{y^2+b^2}^{-\frac{3}{2}} \times y^2 = \frac{2y\dot{y}}{\overline{y^2+b^2}^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}} =$ $\frac{2y\dot{y} \times \overline{y^2+b^2} - y^3\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}} = \frac{y^3\dot{y} + 2b^2y\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}}$; hence $CO = y\dot{y} \times$ $\frac{\overline{y^2+b^2}^{\frac{3}{2}}}{y^3\dot{y} + 2b^2y\dot{y}} = \frac{\overline{y^2+b^2}^{\frac{3}{2}}}{y^2+2b^2}$.

102. The ſame expreſſion for the radius of curvature will do for all curves, where the relation between SY and SC are known. For example, let the curve be a *parabola*, S the focus, and $a = \frac{1}{4}$ the principal latus rectum; then $y = \frac{v^2}{a}$, and $y^2 = \frac{v^4}{a^2}$, $\therefore y\dot{y} = \frac{2v^3\dot{v}}{a^2}$; hence $CO = \frac{y\dot{y}}{\dot{v}} = \frac{2v^3}{a^2}$.

Alſo, as $CL = \frac{4v^3}{a^2}$, we have $y : v :: \frac{4v^3}{a^2} : CV = \frac{4v^4}{a^2y}$ $=$ (as $\frac{v^4}{a^2} = y^2$) $4y = 4CS$.

SECTION

SECTION IX.

ON LOGARITHMS.

Prop. XLV.

GIVEN a number to find it's logarithm.

103. Let $1+x$ be the number, y it's logarithm, and m the modulus; then (Art. 44.) $\dot{y} = \frac{m\dot{x}}{1+x} = m \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + \&c.}$ hence by taking the fluents, $y = m \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.}$ which wants no correction, becauſe when $x=0$, y vaniſhes as it ought, for then the number becomes 1, whoſe log. $=0$. Now this ſeries will converge the quicker the ſmaller x is. If $x=1$, $y = m \times \overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.}$ = the log. of 2. If $m=1$, $y = 1 - \frac{1}{2} + \frac{1}{3} - \&c.$ the h. l. of 2. Hence, as we are at liberty to aſſume m what we pleaſe, we may, to the ſame number, have as many different ſyſtems of logarithms as we pleaſe.

104. But to find a ſeries which ſhall converge quicker, let the given number be $\frac{1+x}{1-x}$; then (Art. 44.) $\dot{y} = 2m \times \frac{\dot{x}}{1-x^2} = 2m \times \overline{\dot{x} + x^2\dot{x} + x^4\dot{x} + \&c.}$ whoſe fluent is $y = 2m \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ If $m = 1$, we get $y = 2 \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ for the hyp. log. of $\frac{1+x}{1-x}$. Let $x = \frac{1}{3}$, then the number becomes 2; hence

$$\left.\begin{array}{rr} x = & 0{,}33333333 \\ \frac{1}{3}x^3 = & 1234567 \\ \frac{1}{5}x^5 = & 82307 \\ \frac{1}{7}x^7 = & 6532 \\ \frac{1}{9}x^9 = & 564 \\ \frac{1}{11}x^{11} = & 51 \\ \hline & 0{,}34657354 \\ & 2 \\ \hline & 0{,}69314708 \\ \hline \end{array}\right\}$$

This h. l. of 2 is true to 6 places; the true value to 7 places being 0,6931472; and it would have required at leaſt 100000 terms of the ſeries in Art. 103. to have given the value to the ſame degree of accuracy.

The common log. of 2 is 0,3010300. Now theſe different values depend on the different values of m, and in the former caſe $m = 1$; hence 0,6931472 : 0,3010300 :: 1 : m in the latter caſe = ,43424948 the modulus of the common ſyſtem. Hence if any common log. be *divided* by this modulus it gives the correſponding hyp. log. Or if any hyp. log. be *multiplied* by it, it gives the correſponding common logarithm. For the various methods which have been invented to calculate logarithms, the reader is referred to Dr. HUTTON's very excellent Introduction to his Tables of Logarithms, and to Mr. MASERES *Scriptores Logarithmici.*

105. By Art. 42. a ſet of quantities A^0, A^1, A^2, A^3, A^4, &c. in geometric progreſſion will have their logarithms in arithmetic progreſſion; hence the indices 0, 1, 2, 3, 4, &c. may repreſent the reſpective logarithms. Now in the common ſyſtem of logarithms, $A=10$; hence the logarithms of 10^0, 10^1, 10^2, 10^3, 10^4, &c. or of 1, 10, 100, 1000, 10000, &c. are 0, 1, 2, 3, 4, &c. And if between 10^0 and 10^1, we inſert an infinite number of geometric means, as 10^n, 10^{2n}, 10^{3n}, &c. n being indefinitely ſmall, then ſome of theſe means muſt neceſſarily make up all the intermediate numbers between 1 and 10, as 2, 3, 4, 5, 6, 7, 8, 9, or at leaſt be indefinitely near to them; the indices therefore of ſuch means muſt be the logarithms of theſe numbers; for inſtance, if $10^{rn}=2$, then $rn=$ log. of 2; if $10^{sn}=7$, then $sn=$log. of 7; and ſo for any other number.

DEFINITION.

106. The *meaſure of a ratio* $1:N$ is the number of times which any other aſſumed ratio $1:A$ muſt be taken to make that ratio. Thus, if $N=A^2$, the meaſure of the ratio of $1:A^2$ is 2, that ratio containing 2 ratios of $1:A$.

107. The ratio of 1 : A^2, 1 : A^3, 1 : A^4, &c. contain 2, 3, 4, &c. ratios of 1 : A; hence the indices of A expreſs the number of ratios of 1 : A which that ratio contains; for inſtance, 1 : A^4 contains 4 ratios of 1 : A; hence 4 is the meaſure of the ratio 1 : A^4; alſo the meaſure of the ratio of 1 : A^m is m, that ratio containing m ratios of 1 : A. Now if we put $A=10$, then the meaſure of the ratio of 1 : 10^m is m; but by the laſt article, m is the logarithm of 10^m; hence the logarithm of any number is the meaſure of the ratio of that number to unity. In this ſenſe, logarithms are called the meaſures of ratios, the logarithm of any number N ſhowing how many ratios of 1 : 10 are neceſſary to make the ratio of 1 : N.

108. Hence every ratio 1 : N has ſome certain meaſure; now that ratio whoſe meaſure is m, the modulus of the ſyſtem, is called the *Modular Ratio* by Mr. COTES.

109. If $x=y^n$, then by taking the logarithms of both ſides, log. $x=n\times$ log. y; hence if we have any equation of this form log. $x=n\times$ log. y, then will $x=y^n$.

LEMMA.

110. If $\left\{\begin{matrix} A+Bx+Cx^2+Dx^3+\&c. \\ a+bx+cx^2+dx^3+\&c. \end{matrix}\right\}=0$, or $\overline{A+a}$ $+\overline{B+b}\times x+\overline{C+c}\times x^2+\overline{D+d}\times x^3+\&c.=0$, whatever be the value of x; then muſt $A+a=0$, $B+b=0$, $C+c=0$, &c. For as we may take x of any value, let $x=0$, and then $A+a=0$; hence the remaining part, $\overline{B+b}\times x+\overline{C+c}\times x^2+\overline{D+d}\times x^3+\&c.=0$, and dividing by x, $\overline{B+b}+\overline{C+c}\times x+\overline{D+d}\times x^2+\&c.=0$; let $x=0$, and then $B+b=0$; and thus we may

proceed for all the coefficients. Or we may conſider it thus: The equation cannot become $=0$ but when it's roots are ſubſtituted for x; the equation therefore cannot vaniſh for *every* value of x you may aſſume, unleſs you make every term vaniſh, independent of x, by making each coefficient $=0$.

Prop. XLVI.

Given a logarithm to find it's number.

111. Let $1+x$ be any number and y it's logarithm, then $\dot{y}=\frac{m\dot{x}}{1+x}$, hence $\dot{y}+x\dot{y}=m\dot{x}$, and $\dot{y}+x\dot{y}-m\dot{x}=0$. Aſſume $x=ay+by^2+cy^3+$ &c. then $\dot{x}=a\dot{y}+2by\dot{y}+3cy^2\dot{y}+$ &c. ſubſtitute theſe values of x and $\dot{x}$ into $\dot{y}+x\dot{y}-m\dot{x}=0$, and we have,

$$\left.\begin{array}{l}\dot{y}+ \quad ay\dot{y}+ \quad by^2\dot{y}+\&c.\\ -ma\dot{y}-2mby\dot{y}-3mcy^2\dot{y}-\&c.\end{array}\right\}=0;$$

hence, (Art. 110.) $1-ma=0$, $a-2mb=0$, $b-3mc=0$, &c. therefore $a=\frac{1}{m}$; $b=\frac{a}{2m}=\frac{1}{2m^2}$; $c=\frac{b}{3m}=\frac{1}{2.3m^3}$; &c. hence $x=\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+$ &c. conſequently $1+x=1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+$ &c. the number whoſe logarithm is y.

If $m=1$, then $1+x=1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+$ &c. is the number whoſe h. l. is y.

PROP. XLVII.

To find the modular ratio.

112. By Art. 107. every logarithm is the meaſure of the ratio of it's correſponding number to 1; hence y is the meaſure of the ratio of $1 + \frac{y}{m} + \frac{y^2}{2m^2} + \frac{y^3}{2.3m^3}$ + &c. to 1; now (Art. 108.) the modular ratio is that ratio of which the modulus is the meaſure; hence if we make $m = y$, m will become the meaſure of the above ratio, and the ratio will become the modular ratio; therefore make $m = y$, and the ratio becomes $1 + 1 + \frac{1}{2} + \frac{1}{2.3}$ + &c. to 1 for the modular ratio, which is therefore the ſame for every ſyſtem, it being independant both of m and y.

SECTION X.

ON THE FLUENTS OF QUANTITIES.

PROP. XLVIII.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n+z^n} = \dot{F}$.

113. Put $a^n = b^2$, $z^n = x^2$, then $z^{\frac{1}{2}n} = x$, ∴ $\frac{n}{2} \times z^{\frac{1}{2}n-1}\dot{z} = \dot{x}$, and $z^{\frac{1}{2}n-1}\dot{z} = \frac{2}{n} \times \dot{x}$; hence $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2+x^2} = \frac{2}{nb^2} \times \frac{b^2\dot{x}}{b^2+x^2}$; conſequently (Art. 46.) $F = \frac{2}{nb^2} \times$ cir. arc whoſe rad. $= b$ tan. $= x$.

PROP. XLIX.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n-z^n} = F$.

PROP. XLVII.

To find the modular ratio.

112. By Art. 107. every logarithm is the meaſure of the ratio of it's correſponding number to 1; hence y is the meaſure of the ratio of $1 + \frac{y}{m} + \frac{y^2}{2m^2} + \frac{y^3}{2.3m^3}$ $+$&c. to 1; now (Art. 108.) the modular ratio is that ratio of which the modulus is the meaſure; hence if we make $m = y$, m will become the meaſure of the above ratio, and the ratio will become the modular ratio; therefore make $m = y$, and the ratio becomes $1 + 1 + \frac{1}{2} + \frac{1}{2.3}$ + &c. to 1 for the modular ratio, which is therefore the ſame for every ſyſtem, it being independant both of m and y.

SECTION X.

ON THE FLUXIONS OF EXPONENTIALS.

DEFINITION.

113. A Quantity is called an *exponential*, when it's index is variable.

PROP. XLVIII.

To find the fluxion of x^y.

114. Put $x^y = z$, and let $X =$ h. l. x; then by the nature of logarithms $yX = Z$, hence $y\dot{X} + X\dot{y} = \dot{Z}$; but by Art. 45. $\dot{X} = \frac{\dot{x}}{x}$, and $\dot{Z} = \frac{\dot{z}}{z}$; hence $\frac{y\dot{x}}{x} + X\dot{y} = \frac{\dot{z}}{z}$, conſequently $\dot{z} = \frac{zy\dot{x}}{x} + zX\dot{y} = yx^{y-1}\dot{x} + Xx^{y}\dot{y}$.

If x be conſtant, then $\dot{x} = 0$, and $\dot{z} = Xx^{y}\dot{y}$.

If y be conſtant, $\dot{y} = 0$, and $\dot{z} = yx^{y-1}\dot{x}$, as in Art. 11.

PROP. XLIX.

To find the fluxion of x^{y^z}.

115. Put $x^{y^z} = w$, and let $x^y = v$, then $v^z = w$; hence if $V =$ h. l. v, we have (Art. 114.) $\dot{w} = zv^{z-1}\dot{v} + Zv^z\dot{z}$; but $v = x^y$, and $\dot{v} = yx^{y-1}\dot{x} + Xx^y\dot{y}$; hence by ſubſtitution, $\dot{w} = zx^{y^z-1} \times \overline{yx^{y-1}\dot{x} + Xx^y\dot{y}} + Zx^{y^z}\dot{z}$ $= zyx^{y^z-1} \times x^{y-1}\dot{x} + zXx^{y^z-1} \times x^y\dot{y} + Zx^{y^z}\dot{z}$. If any one of theſe become conſtant, it's fluxion $= 0$, and the term vaniſhes where that fluxion enters. In like manner we may find the fluxion, whatever be the number of quantities. The meaning of this notation is, the z power of x^y, not the y^z power of x. If this latter had been the meaning of the notation, we muſt have put $y^z = v$, inſtead of $x^y = v$.

ON THE FLUENTS OF QUANTITIES.

Prop. L.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n + z^n} = \dot{F}$.

116. Put $a^n = b^2$, $z^n = x^2$, then $z^{\frac{1}{2}n} = x$, $\therefore \frac{n}{2} \times z^{\frac{1}{2}n-1}\dot{z} = \dot{x}$, and $z^{\frac{1}{2}n-1}\dot{z} = \frac{2}{n} \times \dot{x}$; hence $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2 + x^2}$ $= \frac{nb^2}{2} \times \frac{b^2\dot{x}}{b^2 + x^2}$; conſequently (Art. 46.) $F = \frac{2}{nb^2} \times$ cir. arc, whoſe rad. $= b$, tan. $= x$.

Prop. LI.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n - z^n} = \dot{F}$.

117. By the ſame ſubſtitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2 - x^2} = \frac{1}{nb} \times \frac{2b\dot{x}}{b^2 - x^2}$; hence (Art. 45.) $F = \frac{1}{nb} \times \text{h. l. } \frac{b+x}{b-x}$.

Prop. LII.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a^n + z^n}}$, *to find F.*

118. By the ſame ſubſtitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2 + x^2}}$; hence (Art. 45.) $F = \frac{2}{n} \times \text{h. l. } \overline{x + \sqrt{b^2 + x^2}}$.

Prop. LIII.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a^n - z^n}}$, *to find F.*

119. By the ſame ſubſtitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2 - x^2}} = \frac{2}{nb} \times \frac{b\dot{x}}{\sqrt{b^2 - x^2}}$; hence (Art. 46.) $F = \frac{2}{nb} \times$ cir. arc, rad. $= b$, ſine x.

Prop. LIV.

Let $\dot{F} = \frac{\dot{z}}{\sqrt{az^2 + bz + c}}$, *to find F.*

120. $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\dot{z}}{\sqrt{z^2 + \frac{b}{a} \times z + \frac{c}{a}}}$; put $z + \frac{b}{2a} = x$,

then $z^2+\frac{b}{a}z+\frac{b^2}{4a^2}=x^2$, hence $z^2+\frac{b}{a}z+\frac{c}{a}=x^2-\frac{b^2}{4a^2}$ $+\frac{c}{a}=$ (by putting $\frac{c}{a}-\frac{b^2}{4a^2}=d^2$) x^2+d^2; alſo $\dot{z}=\dot{x}$; hence $\dot{F}=\frac{1}{\sqrt{a}}\times\frac{\dot{x}}{\sqrt{x^2+d^2}}$; ∴ by Art. 45. $F=\frac{1}{\sqrt{a}}$ $\times$ h. l. $\overline{x+\sqrt{x^2+d^2}}$.

PROP. LV.

Let $\dot{F}=\frac{x^{n-1}\dot{x}}{\sqrt{ax^{2n}+bx^n+c}}$, *to find* F.

121. Put $x^n=z$, then $x^{n-1}\dot{x}=\frac{1}{n}\times\dot{z}$, alſo $x^{2n}=z^2$; hence $\dot{F}=\frac{1}{n}\times\frac{\dot{z}}{\sqrt{az^2+bz+c}}$, whoſe fluent is given in the laſt article.

PROP. LVI.

Let $\dot{F}=\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, *to find* F.

122. Aſſume $\dot{v}=\frac{\dot{x}}{\sqrt{a^2+x^2}}$, then (Art. 45.) $v=$ h.l. $\overline{x+\sqrt{a^2+x^2}}$; put $w=\sqrt{a^2x^2+x^4}$, then $\dot{w}=$ $\frac{a^2x\dot{x}+2x^3\dot{x}}{\sqrt{a^2x^2+x^4}}=\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}+\frac{2x^2\dot{x}}{\sqrt{a^2+x^2}}=a^2\dot{v}+2\dot{F}$, hence $\dot{F}=\frac{1}{2}\dot{w}-\frac{1}{2}a^2\dot{v}$, ∴ $F=\frac{1}{2}w-\frac{1}{2}a^2v$. Call this P.

PROP. LVII.

Let $\dot{F}=\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$, *to find* F.

123. Aſſume $\dot{v}=\frac{a\dot{x}}{\sqrt{a^2-x^2}}$, then (Art. 46.) $v=$ cir. arc, rad. $=a$, ſin. $=x$; put $w=\sqrt{a^2x^2-x^4}$, then $\dot{w}=\frac{a^2x\dot{x}-2x^3\dot{x}}{\sqrt{a^2x^2-x^4}}=\frac{a^2\dot{x}}{\sqrt{a^2-x^2}}-\frac{2x^2\dot{x}}{\sqrt{a^2-x^2}}=a\dot{v}-2\dot{F}$, hence $\dot{F}=\frac{1}{2}a\dot{v}-\frac{1}{2}\dot{w}$, $\therefore$ $F=\frac{1}{2}av-\frac{1}{2}w$. Call this $\mathfrak{Q}$.

Prop. LVIII.

Let $\dot{F}=\frac{x^4\dot{x}}{\sqrt{a^2+x^2}}$, *to find* F.

124. Aſſume $v=\sqrt{a^2x^6+x^8}$, then $\dot{v}=\frac{3a^2x^5\dot{x}+4x^7\dot{x}}{\sqrt{a^2x^6+x^8}}=\frac{3a^2x^2\dot{x}}{\sqrt{a^2+x^2}}+\frac{4x^4\dot{x}}{\sqrt{a^2+x^2}}=3a^2\dot{P}$ (Art. 119.) $+4\dot{F}$, hence $\dot{F}=\frac{1}{4}\dot{v}-\frac{3a^2}{4}\dot{P}$, and $F=\frac{1}{4}v-\frac{3a^2}{4}P$.

Prop. LIX.

Let $\dot{F}=\frac{x^4\dot{x}}{\sqrt{a^2-x^2}}$, *to find* F.

125. Aſſume $v=\sqrt{a^2x^6-x^8}$, then $\dot{v}=\frac{3a^2x^5\dot{x}-4x^7\dot{x}}{\sqrt{a^2x^6-x^8}}$ $=\frac{3a^2x^2\dot{x}}{\sqrt{a^2-x^2}}-\frac{4x^4\dot{x}}{\sqrt{a^2-x^2}}=3a^2\dot{\mathfrak{Q}}$ (Art. 120.) $-4\dot{F}$, hence $\dot{F}=\frac{3a^2}{4}\dot{\mathfrak{Q}}-\frac{1}{4}\dot{v}$, and $F=\frac{3a^2}{4}\mathfrak{Q}-\frac{1}{4}v$.

In this manner we may continue the fluents when the numerators are $x^6\dot{x}$, $x^8\dot{x}$, $x^{10}\dot{x}$, &c. by aſſuming

$v=$

$v=\sqrt{a^2x^{10}\pm x^{12}}$, $\sqrt{a^2x^{14}\pm x^{16}}$, $\sqrt{a^2x^{18}\pm x^{20}}$, &c. reſpectively, and by taking the fluxion you will, in like manner, get $\dot{v}$ in terms of the given fluxion and of the next inferior fluxion.

PROP. LX.

Let $\dot{F}=x^n\dot{x}\sqrt{a^2\pm x^2}$, *n being an even number, to find F.*

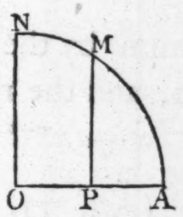
126. Multiply and divide the fluxion by $\sqrt{a^2\pm x^2}$ and $\dot{F}=\dfrac{a^2x^n\dot{x}\pm x^{n+2}\dot{x}}{\sqrt{a^2\pm x^2}}$; hence as the indices of x in the numerator are even numbers, the fluents of $\dfrac{a^2x^n\dot{x}}{\sqrt{a^2\pm x^2}}$, and $\dfrac{x^{n+2}\dot{x}}{\sqrt{a^2\pm x^2}}$, may each be found by the method directed in the laſt article.

If n be an odd number, F may be found by Art. 41.

PROP. LXI.

Let $\dot{F}=\dot{x}\sqrt{2ax-x^2}$, *to find F.*

127. Let the radius $AO=a$, $AP=x$, then the ſine



$PM=\sqrt{2ax-x^2}$, $\therefore\ \dot{F}=\dot{x}\sqrt{2ax-x^2}=$ (Art. 49.) the fluxion of the area APM; hence $F=$ area APM.

PROP. LXII.

Let $\dot F = x\dot x\sqrt{2ax - x^2}$, *to find F.*

128. Aſſume $w = \frac{1}{3} \times \overline{2ax - x^2}^{\frac{3}{2}}$, then $\dot w = \overline{a\dot x - x\dot x} \times \sqrt{2ax - x^2} = a\dot x\sqrt{2ax - x^2} - \dot F$, hence $\dot F = a\dot x\sqrt{2ax - x^2} - \dot w$, $\therefore F = a \times$ area $APM - w$.

PROP. LXIII.

Let $\dot F = \frac{x\dot x}{\sqrt{2ax - x^2}}$, *to find F.*

129. Aſſume $w = \sqrt{2ax - x^2}$, then $\dot w = \frac{a\dot x - x\dot x}{\sqrt{2ax - x^2}} = \frac{a\dot x}{\sqrt{2ax - x^2}} - \frac{x\dot x}{\sqrt{2ax - x^2}} = \frac{a\dot x}{\sqrt{2ax - x^2}} - \dot F$, hence $\dot F = \frac{a\dot x}{\sqrt{2ax - x^2}} - \dot w$, $\therefore$ (Art. 46.) $F = z - w$, z being a cir.arc, rad. $= a$, verſed ſine $= x$.

PROP. LXIV.

Let $\dot F = \frac{x^m\dot x}{x - a}$, *to find F.*

130. Divide the num. by the den. till the index of x in the remainder $= 0$, and the remainder will then be $a^m\dot x$; hence $\dot F = x^{m-1}\dot x + ax^{m-2}\dot x + a^2x^{m-3}\dot x + \&c. + a^m \times \frac{\dot x}{x - a}$; therefore (Art. 37. and 45.) $F = \frac{x^m}{m} + \frac{ax^{m-1}}{m-1} + \frac{a^2x^{m-2}}{m-2} + \&c. + a^m \times \text{h. l. } \overline{x - a}$. Here m muſt

be a whole poſitive number, otherwiſe the index of x cannot become $=0$. If the denominator be $x+a$, the terms will be alternately $+$ and $-$.

Prop. LXV.

Let $\dot F = \dfrac{z^{rm-1}\dot z}{a+bz^m}$, *to find F.*

131.
$$\begin{array}{rl}
bz^m+a\big) & z^{rm-1}\dot z \qquad \Big(\frac{1}{b}\times z^{rm-m-1}\dot z - \frac{a}{b^2}\times z^{rm-2m-1}\dot z + \&c. \\
 & \underline{z^{rm-1}\dot z + \frac{a}{b}\times z^{rm-m-1}\dot z} \\
 & \qquad -\frac{a}{b}\times z^{rm-m-1}\dot z \\
 & \qquad \underline{-\frac{a}{b}\times z^{rm-m-1}\dot z - \frac{a^2}{b^2}\times z^{rm-2m-1}\dot z} \\
 & \qquad\qquad\qquad\qquad \frac{a^2}{b^2}\times z^{rm-2m-1}\dot z \\
 & \qquad\qquad\qquad\qquad \&c. \qquad \&c.
\end{array}$$

continue this diviſion till the index of z in the remainder becomes $m-1$, and the remainder will be $\pm\frac{a^{r-1}}{b^{r-1}}\times z^{m-1}\dot{z}$; hence $\dot{F}=\frac{1}{b}\times z^{rm-m-1}\dot{z}-\frac{a}{b^2}\times z^{rm-2m-1}\dot{z}+\&c.\pm\frac{a^{r-1}}{b^{r-1}}\times\frac{z^{m-1}\dot{z}}{a+bz^m}$; now the laſt term $=\pm\frac{a^{r-1}}{mb^r}\times\frac{mbz^{m-1}\dot{z}}{a+bz^m}$; hence (Art. 37. and 45.) $F=\frac{1}{b}\times\frac{z^{rm-m}}{rm-m}-\frac{a}{b^2}\times\frac{z^{rm-2m}}{rm-2m}+\&c.\pm\frac{a^{r-1}}{mb^r}\times$ h.l. $\overline{a+bz^m}$. Here r muſt be a whole poſitive number, otherwiſe the index of z can never become $m-1$.

LEMMA.

Let $\frac{1}{x^n-px^{n-1}+\&c.}=\frac{K}{x-a}+\frac{L}{x-b}+\frac{M}{x-c}+\&c.$ to find K, L, M, &c. where a, b, c, &c. are the roots of $x^n-px^{n-1}+\&c.+0$.

132. Reduce the fractions to a common denominator, and it will be the ſame as the denominator on the left, and conſequently the ſum of the numerators $=1$; hence $K\times\overline{x-b}\times\overline{x-c}\times\&c.+L\times\overline{x-a}\times\overline{x-c}\times\&c.+M\times\overline{x-a}\times\overline{x-b}\times\&c.+\&c.=1$; now as this is true let x be what it will, make $x=a$, and then $K\times\overline{a-b}\times\overline{a-c}\times\&c.=1$, $\therefore K=\frac{1}{\overline{a-b}\times\overline{a-c}\times\&c.}$. Make $x=b$, and then $L\times\overline{b-a}\times\overline{b-c}\times\&c.=1$, $\therefore$ $L=\frac{1}{\overline{b-a}\times\overline{b-c}\times\&c.}$. In like manner we get the other numerators.

PROP.

PROP. LXVI.

Let $\dot{F} = \frac{x^m\dot{x}}{x^n - px^{n-1} + \&c.}$, *to find F, m being a whole poſitive number.*

133. Let $\frac{1}{x^n - px^{n-1} + \&c.} = \frac{K}{x-a} + \frac{L}{x-b} + \&c.$ then K, L, &c. are known by the laſt article; hence $\frac{x^m\dot{x}}{x^n - px^{n-1} + \&c.} = \frac{Kx^m\dot{x}}{x-a} + \frac{Lx^m\dot{x}}{x-b} + \&c.$ Now (Art. 127.) the fluent of $\frac{Kx^m\dot{x}}{x-a}$ is $\frac{Kx^m}{m} + \frac{Kax^{m-1}}{m-1} + \&c. + Ka^m \times \text{h. l. } \overline{x-a}$; in like manner the fluents of all the other quantities are found, the ſum of all which is F. Now the ſum of all theſe quantities $= \overline{K+L+\&c.} \times \frac{x^m}{m} + \overline{Ka+Lb+\&c.} \times \frac{x^{m-1}}{m-1} + \&c. + Ka^m \times \text{h. l. } \overline{x-a} + Lb^m \times \text{h. l. } \overline{x-b} + \&c.$ But by DR. WARING'S *Med. Alg. laſt edit. in the Addenda*, $K+L+\&c. = 0$; $Ka+Lb+\&c. = 0$; &c. through all thoſe terms, when m is leſs than n; in this caſe therefore $F = Ka^m \times \text{h. l. } \overline{x-a} + L\,b^m \times \text{h.l. } \overline{x-b} + \&c.$ If m be equal to or greater than n, the coefficients of the firſt $n-1$ terms will become $= 0$.

134. If m be leſs than n, the quantity $\frac{x^m\dot{x}}{x^n - px^{n-1} + \&c.}$ may be reſolved into $\frac{\overset{1}{K}\dot{x}}{x-a} + \frac{\overset{1}{L}\dot{x}}{x-b} + \frac{\overset{1}{M}\dot{x}}{x-c} + \&c.$ for in this caſe $\overset{1}{K} \times \overline{x-b} \times \overline{x-c} \times \&c. + \overset{1}{L} \times \overline{x-a} \times \overline{x-c} \times \&c. + \&c. = x^m$; hence if $x = a$, $\overset{1}{K} = \frac{a^m}{\overline{a-b} \times \overline{a-c} \times \&c}$;

if $x=b$, $\overset{1}{L}=\frac{b^m}{\overline{b-a}\times\overline{b-c}\times\&c.}$; &c. The reaſon why m muſt be leſs than n is this: The equation $\overset{1}{K}\times\overline{x-b}\times\overline{x-c}\times\&c.+\overset{1}{L}\times\overline{x-a}\times\overline{x-c}\times\&c.+\&c.+\&c.-x^m=0$; and that this may be *always* true, the coefficients of the like powers of x muſt be aſſumed $=0$ (Art. 110.), and by ſuch an aſſumption you would deduce the ſame values of K, L, &c. as above. Now the product of each of the quantities into which K, L, &c. are multiplied is of $n-1$ dimenſions in terms of x, there being $n-1$ factors; hence if m be greater than $n-1$, there is only the term in which x is of m dimenſions, therefore this term can never be made to vaniſh generally along with the reſt. But if m be equal to or leſs than $n-1$, then this term x^m will come in with others having the ſame power, and the whole coefficient will be made $=0$.

But the denominators may be otherwiſe expreſſed; for as $\overline{x-a}\times\overline{x-b}\times\&c.=x^n-px^{n-1}+\&c$ $\therefore$ by taking the fluxion we have $\dot{x}\times\overline{x-b}\times\overline{x-c}\times\&c.+\dot{x}\times\overline{x-a}\times\overline{x-c}\times\&c.+\&c.=nx^{n-1}\dot{x}-\overline{n-1}.px^{n-2}\dot{x}+\&c.$ hence if $x=a$, we have $\overline{a-b}\times\overline{a-c}\times\&c.=na^{n-1}-\overline{n-1}.pa^{n-2}+\&c.$ If $x=b$, then $\overline{b-a}\times\overline{b-c}\times\&c.=nb^{n-1}-\overline{n-1}.pb^{n-2}+\&c.$ and ſo on for the reſt; hence take the fluxion of the given equation, omitting $\dot{x}$, and write a, b, c, &c. for x and you get the denominators.

Hence when m is leſs than n, the fluent of $\frac{x^m\dot{x}}{x^n-px^{n-1}+\&c.}$ is $\overset{1}{K}\times$ h. l. $\overline{x-a}+\overset{1}{L}\times$ h. l. $\overline{x-b}+$

&c.

&c. which agrees with the concluſion in Art. 130. becauſe $\overset{1}{K}=Ka^m$, $\overset{1}{L}=Lb^m$, &c.

135. If two roots a, b, be equal, one of the quantities muſt have a quadratic diviſor $\overline{x-a}^2$. For example: Let $\frac{1}{x^3-px^2+qx-r}=\frac{Lx+M}{\overline{x-a}^2}+\frac{N}{x-c}$: then reducing the two quantities on the right to the ſame denominator, and making the numerators equal, we get $Lx^2-Lcx+Mx-Mc+Nx^2-2Nax+Na^2-1=0$, hence (Art. 110.) making $L+N=0$, $M-Lc-2Na=0$, $-Mc+Na^2-1=0$, we have, $L=-N$, $M=\frac{Na^2-1}{c}$; conſequently $\frac{Na^2-1}{c}+Nc-2Na=0$, therefore $N=\frac{1}{\overline{a-c}|^2}$; $L=\frac{1}{-\overline{c-a}|^2}$; $M\frac{2a-c}{\overline{a-c}|^2}$. Hence the fluent of $\frac{\dot{x}}{x^3-px^2+qx-r}$, or $\frac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2}+\frac{N\dot{x}}{x-c}$ may be thus found. Put $x-a=z$, then $x=z+a$, and $\dot{x}=\dot{z}$; hence $\frac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2}=\frac{Lz\dot{z}+La\dot{z}+M\dot{z}}{z^2}=$ (if $La+M=b$) $\frac{L\dot{z}}{z}+\frac{b\dot{z}}{z^2}$, whoſe fluent (Art. 45. and 37.) is $L\times$ h. l. $z-\frac{b}{z}=$ $L\times$ h. l. $\overline{x-a}-\frac{b}{x-a}$; and the fluent of $\frac{N\dot{x}}{x-c}$ is $N\times$ h. l. $\overline{x-c}$.

136. If two of the roots be impoſſible, thoſe two binomial fractions muſt be incorporated into one. If

1

$\frac{1}{x^3-px^2+qx-r}=\frac{L}{x-a}+\frac{M}{x-b}+\frac{N}{x-c}$, and a and b be impoſſible, then make $\frac{L}{x-a}+\frac{M}{x-b}=\frac{\overline{L+M}\times x-\overline{Lb+Ma}}{x^2-\overline{a+b}\times x+ab}$, where the impoſſible quantities vaniſh, as will appear by ſubſtituting $m+n\sqrt{-1}$ for a, and $m-n\sqrt{-1}$ for b.

PROP. LXVII.

Let $\dot{F}=\frac{cx\dot{x}+d\dot{x}}{x^2-px+q}$, *to find F.*

137. Put $x-\frac{1}{2}p=z$, then $x=z+\frac{1}{2}p$, alſo $\dot{x}=\dot{z}$; hence $d\dot{x}=d\dot{z}$, and $cx\dot{x}=cz\dot{z}+\frac{1}{2}pc\dot{z}$, $\therefore$ $cx\dot{x}+d\dot{x}=cz\dot{z}+\overline{\frac{1}{2}pc+d}\times\dot{z}=$ (if $\frac{1}{2}pc+d=e$) $cz\dot{z}+e\dot{z}$; alſo $x^2-px+\frac{1}{4}p^2=z^2$, hence $x^2-px+q=z^2+q-\frac{1}{4}p^2=$ (if $q-\frac{1}{4}p^2=a^2$) $z^2\pm a^2$ according as a^2 is poſitive or negative, or according as the two values of x are impoſſible or poſſible. Hence $\dot{F}=\frac{cz\dot{z}+e\dot{z}}{z^2\pm a^2}=\frac{cz\dot{z}}{z^2\pm a^2}+\frac{e\dot{z}}{z^2\pm a^2}$. Now (Art. 45.) the fluent of $\frac{cz\dot{z}}{z^2\pm a^2}$ is $\frac{1}{2}c\times$ h. l. $\overline{z^2\pm a^2}$. Alſo, taking $+a^2$, $\frac{e\dot{z}}{z^2+a^2}=\frac{e}{a^2}\times\frac{a^2\dot{z}}{z^2+a^2}$, whoſe fluent (Art. 46.) is $\frac{e}{a^2}\times$ cir. arc, rad. $=a$, tan. $=z$. But taking $-a^2$, $\frac{e\dot{z}}{z^2-a^2}=-\frac{e}{2a}\times\frac{2a\dot{z}}{a^2-z^2}$, whoſe fluent (Art. 45.) is $-\frac{e}{2a}\times$ h. l. $\frac{a+z}{a-z}$;

call the fluent of this ſecond part B, and $F=\frac{1}{2}c\times$ h. l. $\overline{z^2\pm a^2}+B$. Call this fluent $\mathcal{Q}$.

PROP. LXVIII.

Let $\dot{F}=\frac{z^m\dot{z}}{z^2-pz+q}$, *to find F.*

138. If the roots be both poſſible, then (Art. 129) reſolve $\frac{1}{z^2-pr+q}$ into $\frac{K}{z-a}+\frac{L}{z-b}$; and $\dot{F}=\frac{Lz^m\dot{z}}{z-a}+\frac{Mz^m\dot{z}}{z-b}$, whoſe fluents are found (Art. 127.) But if the roots be impoſſible, divide $z^m\dot{z}$ by z^2-pz+q until the remainder becomes $cz\dot{z}+e\dot{z}$, c and e being put for the coefficients which ariſe from the diviſion, and let the quotient be $z^{m-2}\dot{z}+az^{m-3}\dot{z}+bz^{m-4}\dot{z}+$&c. where $a=p$, $b=p^2-q$ &c. hence $\dot{F}=z^{m-2}\dot{z}+az^{m-3}\dot{z}+bz^{m-4}\dot{z}+$ &c. $+\frac{cz\dot{z}+e\dot{z}}{z^2-pz+q}$, conſequently (Art. 37. and 134.) $F=\frac{z^{m-1}}{m-1}+\frac{az^{m-2}}{m-2}+\frac{bz^{m-3}}{m-3}+$&c.$+\mathcal{Q}$.

If $m=2$, then $F=z+\mathcal{Q}$.
If $m=3$, then $F=\frac{1}{2}z^2+az+\mathcal{Q}$.
If $m=4$, then $F=\frac{1}{3}z^3+\frac{1}{2}az^2+bz+\mathcal{Q}$.

PROP. LXIX.

Let $\dot{F}=\frac{z^{-m}\dot{z}}{z^2-pz+q}$, *to find F.*

139. Put $x=\frac{1}{z}=z^{-1}$, then $x^{m-1}=z^{-m+1}$, and $x^{m-2}\dot{x}$

=

$= -z^{-m}\dot{z}$; hence $\dot{F} = \dfrac{-x^{m-2}\dot{x}}{\frac{1}{x^2} - \frac{p}{x} + q} = -\dfrac{x^m\dot{x}}{1 - px + qx^2}$

$= -\dfrac{1}{q} \times \dfrac{x^m\dot{x}}{\frac{1}{q} - \frac{px}{q} + x^2} =$ (if $\frac{1}{q} = q'$, $\frac{p}{q} = p'$) $-\dfrac{1}{q} \times \dfrac{x^m\dot{x}}{x^2 - p'x + q'}$, which is the ſame as the laſt form.

Prop. LXX.

Let $\dot{F} = \dfrac{\dot{z}}{z\sqrt{a + cz^n}}$, *to find F.*

140. Firſt, $\dot{F} = \dfrac{1}{\sqrt{c}} \times \dfrac{\dot{z}}{z\sqrt{d^2 + z^n}}$ $\left(\text{putting } d^2 = \dfrac{a}{c}\right)$; put $z^{\frac{1}{2}n} = x$, and then $z^n = x^2$; alſo $\dfrac{\dot{x}}{x} = \dfrac{\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}}{z^{\frac{1}{2}n}} = \dfrac{1}{2}n \times \dfrac{\dot{z}}{z}$, $\therefore \dfrac{2}{n} \times \dfrac{\dot{x}}{x} = \dfrac{\dot{z}}{z}$; hence $\dot{F} = \dfrac{2}{n\sqrt{c}} \times \dfrac{\dot{x}}{x\sqrt{d^2 + x^2}}$ $= \dfrac{1}{nd\sqrt{c}} \times \dfrac{2d\dot{x}}{x\sqrt{d^2 + x^2}}$; and (Art. 45.) $F = \dfrac{1}{nd\sqrt{c}} \times \text{h. l. } \dfrac{d - \sqrt{d^2 + x^2}}{d + \sqrt{d^2 + x^2}}$. If d^2 be negative, $\dot{F} = \dfrac{2}{n\sqrt{c}} \times \dfrac{\dot{x}}{x\sqrt{x^2 - d^2}} = \dfrac{2}{nd^2\sqrt{c}} \times \dfrac{d^2\dot{x}}{x\sqrt{x^2 - d^2}}$, and (Art. 46.) $F = \dfrac{2}{nd^2\sqrt{c}} \times$ cir. arc, rad. $= d$, ſecant $= x$.

PROP. LXX.

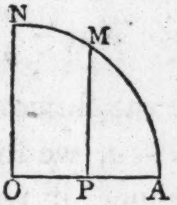
Let $\dot{F} = \dfrac{\dot{z}}{z^2\sqrt{a^2+z^2}}$, *to find F.*

144. Put $x = \dfrac{a^2}{z}$, then $\dot{x} = -\dfrac{a^2\dot{z}}{z^2}$, hence $-\dfrac{1}{a^2} \times \dot{x} = \dfrac{\dot{z}}{z^2}$; therefore $\dot{F} = -\dfrac{1}{a^2} \times \dfrac{\dot{x}}{\sqrt{a^2+\dfrac{a^4}{x^2}}} = -\dfrac{1}{a^3} \times \dfrac{x\dot{x}}{\sqrt{x^2+a^2}}$; hence (Art. 39.) $F = -\dfrac{1}{a^3} \times \sqrt{x^2+a^2}$.

PROP. LXXI.

Let $\dot{F} = \dfrac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$, *to find F.*

145. Put $x = \sqrt{c^2-z^2}$, then $z^2 = c^2 - x^2$, $\therefore z\dot{z} = -x\dot{x}$, and $\sqrt{b^2+z^2} = \sqrt{b^2+c^2-x^2} =$ (if $a^2 = b^2 + c^2$) $\sqrt{a^2-x^2}$; hence $\dot{F} = -\dot{x}\sqrt{a^2-x^2}$; but if $a = OA$, $x = OP$, then $PM = \sqrt{a^2-x^2}$; hence $\dot{F} = -$ the fluxion of the area *OPMN* (Art. 49.), conſequently $F = -$ area *OPMN*.



LEMMA.

To reſolve $\frac{1}{\overline{x+a}^{m}\times\overline{x+b}^{n}}$ into $\frac{H}{\overline{x+a}^{m}}+\frac{K}{\overline{x+a}^{m-1}}$ $+\frac{L}{\overline{x+a}^{m-2}}+$ &c. $+\frac{P}{\overline{x+b}^{n}}+\frac{Q}{\overline{x+b}^{n-1}}+\frac{R}{\overline{x+b}^{n-2}}+$ &c. continued to m and n quantities reſpectively.

146. Reduce the fractions to a common denominator, and make the numerators on each ſide equal, and (A) $H\times\overline{x+b}^{n}+K\times\overline{x+b}^{n}\times\overline{x+a}+L\times\overline{x+b}^{n}\times\overline{x+a}^{2}+$ &c. $+P\times\overline{x+a}^{m}+Q\times\overline{x+a}^{m}\times\overline{x+b}+R\times\overline{x+a}^{m}\times\overline{x+b}^{2}+$ &c. $=1$. Make $x+a=0$, or $x=-a$, and every term, where $x+a$ enter, becomes $=0$, hence $H\times\overline{x+b}^{n}=1$, or $H\times\overline{b-a}^{n}=1$, $\therefore H=\frac{1}{\overline{b-a}^{n}}$. Take the fluxion of equation (A), and omitting $\dot{x}$, we have (B) $nH\times\overline{x+b}^{n-1}+nK\times\overline{x+b}^{n-1}\times\overline{x+a}+K\times\overline{x+b}^{n}$ $+$ &c. $=0$; make $x=-a$, and we have $nH\times\overline{b-a}^{n-1}$ $+K\times\overline{b-a}^{n}=0$, hence $K=-\frac{nH}{b-a}=\frac{-n}{\overline{b-a}^{n+1}}$; thus by continuing to take the fluxion of the laſt equation, and then making $x=-a$, we ſhall get the values of L, &c. In like manner, if we make $x+b=0$, or $x=-b$, then $P=\frac{1}{\overline{a-b}^{m}}$; then by taking the fluxion of the equation and making $x=-b$, we get $Q=$ $\frac{-m}{\overline{a-b}^{m+1}}$; and by proceding as before we get R, &c.

PROP.

PROP. LXXII.

Let $\dot{F} = \frac{x^r\dot{x}}{\overline{x+a}^m \times \overline{x+b}^n}$, *to find F, r being a whole poſitive number.*

147. By the laſt article, $\dot{F} = \frac{Hx^r\dot{x}}{\overline{x+a}^m} + \frac{Kx_r\dot{x}}{\overline{x+a}^{m-1}} +$ &c. $+ \frac{Px^r\dot{x}}{\overline{x+b}^n} + \frac{\mathcal{Q}x^r\dot{x}}{\overline{x+b}^{n-1}} +$ &c. Put $x+a=z$, then $x=z-a$, $\therefore x^{r+1} = \overline{z-a}^{r+1}$, and $x^r\dot{x} = \overline{z-a}^r \times \dot{z}$, hence $\frac{x^r\dot{x}}{\overline{x+a}^m} = \frac{\overline{z-a}^r \times \dot{z}}{z^m} = z^{r-m}\dot{z} - raz^{r-m-1}\dot{z} + r.\frac{r-1}{2}a^2z^{r-m-2}\dot{z} -$ &c. where the number of the terms $=r+1$, and the fluent of every term is found (Art. 37.), except that term where the index of z is -1, whoſe fluent is found by Art. 45. and the ſum of all theſe is the fluent required.

PROP. LXXIII.

Given A the fluent of $\overline{e+fx^n}^m \times x^p\dot{x}$, *to find B the fluent of* $\overline{e+fx^n}^m \times x^{p+n}\dot{x}$, *and C the fluent of* $\overline{e+fx^n}^{m+1} \times x^p\dot{x}$

142. Aſſume $\mathcal{Q} = \overline{e+fx^n}^{m+1} \times x^{p+1}$, then $\dot{\mathcal{Q}} = \overline{p+1} \times \overline{e+fx^n}^{m+1} \times x^p\dot{x} + \overline{m+1} \times nf \times \overline{e+fx^n}^m \times x^{p+n}\dot{x} = \overline{p+1} \times \dot{C} + \overline{m+1} \times nf \times \dot{B}$, hence by taking the fluents, $\mathcal{Q} = \overline{p+1} \times C + \overline{m+1} \times nf \times B$. Alſo $\overline{e+fx^n}^{m+1} \times x^p\dot{x} = \overline{e+fx^n} \times \overline{e+fx^n}^m \times x^p\dot{x} = e \times \overline{e+fx^n}^m \times x^p\dot{x} + f \times \overline{e+fx^n}^m \times x^{p+n}\dot{x}$, that is $\dot{C} = e\dot{A} + f\dot{B}$, therefore

$C = eA + fB$. Now from the *firſt* fluent, $B = \frac{\mathcal{Q} - \overline{p+1} \times C}{\overline{m+1} \times nf}$, and from the *ſecond*, $B = \frac{C - eA}{f}$; hence $\frac{\mathcal{Q} - \overline{p+1} \times C}{\overline{m+1} \times nf} = \frac{C - eA}{f}$, $\therefore C = \frac{\mathcal{Q} + \overline{m+1} \times neA}{p+1+\overline{m+1} \times n}$; conſequently $B = \frac{C - eA}{f} = \frac{C}{f} - \frac{eA}{f} = \frac{\mathcal{Q} + \overline{m+1} \times neA}{\overline{p+1+\overline{m+1} \times n} \times f} - \frac{eA}{f}$. Hence we may continue the fluent as far as we pleaſe, increaſing m by 1, and p by n.

Let $e = a^2$, $f = 1$, $m = -\frac{1}{2}$, $p = 0$, $n = 2$, then $\dot{A} = \frac{\dot{x}}{\sqrt{a^2 + x^2}}$, and $A = \text{h. l. } \overline{x + \sqrt{x^2 + a^2}}$ (Art. 45.), hence B the fluent of $\frac{x^2 \dot{x}}{\sqrt{a^2 + x^2}} = \frac{1}{2} x \times \overline{a^2 + x^2}\big|^{\frac{1}{2}} - \frac{1}{2} a^2 A$ as in Art. 122. alſo C the fluent of $\overline{a^2 + x^2}\big|^{\frac{1}{2}} \times \dot{x} = \frac{1}{2} x \times \overline{a^2 + x^2}\big|^{\frac{1}{2}} + \frac{1}{2} a^2 A$.

Prop. LXXIV.

Let $\dot{F} = v x^n \dot{x}$, *where* $v = h.\, l.\, \frac{1}{1-x}$, *to find* F.

149. Aſſume $\frac{v x^{n+1}}{n+1} + r = F$, then $v x^n \dot{x} + \frac{x^{n+1} \dot{v}}{n+1} + \dot{r} = \dot{F} = v x^n \dot{x}$; hence $\dot{r} = -\frac{x^{n+1} \dot{v}}{n+1} =$ (becauſe $\dot{v} = \frac{\dot{x}}{1-x}$) $= -\frac{x^{n+1} \dot{x}}{\overline{n+1} \times \overline{1-x}} =$ (by diviſion) $-\frac{1}{n+1} \times \overline{-x^n \dot{x} - x^{n-1} \dot{x} - \&c. + \frac{\dot{x}}{1-x}}$, therefore $r = \frac{1}{n+1} \times$

x^{1+n}

$\overline{\frac{x^{n+1}}{n+1}+\frac{x^n}{n}+\&c.-v}$; hence $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\overline{\frac{x^{n+1}}{n+1}+\frac{x^n}{n}+\&c.-v}$.

PROP. LXXV.

Let $\dot{F}=vx^n\dot{x}$, where v is a circular arc whoſe radius is 1 *and tangent x, to find F.*

150. Aſſume $\frac{vx^{n+1}}{n+1}+r=F$; then $vx^n\dot{x}+\frac{x^{n+1}\dot{v}}{n+1}+\dot{r}=\dot{F}=vx^n\dot{x}$. Let *n* be an *odd* number, and then $\dot{r}=-\frac{x^{n+1}\dot{v}}{n+1}=$ (Art. 46.) $-\frac{x^{n+1}\dot{x}}{\overline{n+1}\times\overline{1+x^2}}=-\frac{1}{n+1}\times\overline{x^{n-1}\dot{x}-x^{n-3}\dot{x}+\&c.\pm\dot{v}\left(\frac{\dot{x}}{1+x^2}\right)}$, where the ſign of $\dot{v}$ will be + or − according as $\frac{n+1}{2}$ is even or odd; hence $r=\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x}{n-2}-\&c.\mp v}$; therefore $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v}$. If *n* be an *even* number, the laſt term of the diviſion will be $\pm\frac{x\dot{x}}{1+x^2}$, whoſe fluent is $\pm\frac{1}{2}$ h. l. $\overline{1+x^2}=$ ± h. l. $\sqrt{1+x^2}$; hence $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp \text{h. l. }\sqrt{1+x^2}}$, where the ſign of the laſt term is + or − according as $\frac{1}{2}n$ is odd or even.

PROP. LXXVI.

Let $\dot{F}=z^m x^{n-1}\dot{x}$, *where* $z=h.l.x$, *to find* F.

151. Aſſume $F=az^m+bz^{m-1}+cz^{m-2}+$ &c. a, b, c, &c. being variable coefficients in terms of x; hence by taking the fluxion we have,

$$\left.\begin{array}{l}\dot{a}z^m+\dot{b}z^{m-1}+\qquad\qquad\dot{c}z^{m-2}+\&c.\\ ma\dot{z}z^{m-1}+\overline{m-1}.b\dot{z}z^{m-2}+\&c.\end{array}\right\}=z^m x^{n-1}\dot{x};$$

but by Art. 45. $\dot{z}=\frac{\dot{x}}{x}$; hence by tranſpoſition,

$$\left.\begin{array}{l}\dot{a}z^m+\qquad\dot{b}z^{m-1}+\qquad\qquad\dot{c}z^{m-2}+\&c.\\ -x^{n-1}\dot{x}z^m+\frac{ma\dot{x}}{x}z^{m-1}+\frac{\overline{m-1}.b\dot{x}}{x}z^{m-2}+\&c.\end{array}\right\}=0,$$

$\therefore$ by Art. 110. $\dot{a}-x^{n-1}\dot{x}=0$, $\dot{b}+\frac{ma\dot{x}}{x}=0$, $\dot{c}+\frac{\overline{m-1}.b\dot{x}}{x}=0$, &c. hence $\dot{a}=x^{n-1}\dot{x}$, $\therefore$ $a=\frac{x^n}{n}$; $\dot{b}=\frac{-ma\dot{x}}{x}=\frac{-mx^{n-1}\dot{x}}{n}$, $\therefore$ $b=\frac{-mx^n}{n^2}$; $\dot{c}=\frac{-\overline{m-1}.b\dot{x}}{n}=\frac{-\overline{m-1}\times-mx^{n-1}\dot{x}}{n^2}$, $\therefore$ $c=\frac{m.\overline{m-1}.x^n}{n^3}$; &c. hence $F=\frac{x^n}{n}\times z^m-\frac{mx^n}{n^2}\times z^{m-1}+\frac{m.\overline{m-1}.x^n}{n^3}\times z^{m-2}-$ &c. where the law of continuation is manifeſt, and the ſeries will terminate when m is a whole poſitive number.

PROP. LXXVII.

Let $\dot{F}=a^x x^n\dot{x}$, *to find* F.

152. Aſſume $F=a^x\times\overline{px^n+qx^{n-1}+rx^{n-2}+\&c.}$ and let $m=$ h. l. a; then (Art. 114.) $ma^x\dot{x}$ is the fluxion of a^x; hence by taking the fluxions,

$$\left.\begin{array}{l} ma^x\dot{x}\times\overline{px^n+qx^{n-1}+rx^{n-2}+\&c.} \\ a^x\times\overline{npx^{n-1}\dot{x}+\overline{n-1}.qx^{n-2}\dot{x}+\&c.} \end{array}\right\}=a^xx^n\dot{x};$$

divide both ſides by $a^x\dot{x}$, and tranſpoſe x^n, and we have

$$\left.\begin{array}{r} mpx^n+mqx^{n-1}+mrx^{n-2}+\&c. \\ -x^n+npx^{n-1}+\overline{n-1}.qx^{n-2}+\&c. \end{array}\right\}=0,$$

hence (Art. 110.) $mp-1=0$, $mq+np=0$, $mr+\overline{n-1}.q=0$, &c. $\therefore p=\frac{1}{m}$; $q=\frac{-np}{m}=-\frac{n}{m^2}$; $r=-\frac{\overline{n-1}.q}{m}=-\frac{\overline{n-1}\times-n}{m^3}=\frac{n.\overline{n-1}}{m^3}$; &c. therefore $F=a^x\times\overline{\frac{1}{m}x^n-\frac{n}{m^2}x^{n-1}+\frac{n.\overline{n-1}}{m^3}x^{n-2}-\&c.}$ where the law of continuation is manifeſt, and the ſeries will terminate when n is a whole number.

PROP. LXXVIII.

To find the fluent of $\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}$.

153. Aſſume $\frac{az^{r+1}}{1\pm z^n}+\mathcal{Q}$ for the fluent; then, by taking the fluxion, we have $\frac{\overline{r+1}\times az^r\dot{z}\times\overline{1\pm z^n}\mp naz^{r+n}\dot{z}}{\overline{1\pm z^n}|^2}+\dot{\mathcal{Q}}=\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}$, or $\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}=$

$\frac{1}{1\pm z^n}\times\overline{r+1}\times az^r\dot{z}\mp\frac{naz^{r+n}\dot{z}}{1\pm z^n}+\dot{\mathcal{Q}}$; but $\mp\frac{naz^{r+n}\dot{z}}{1\pm z^n}$ $=-naz^r\dot{z}+\frac{naz^r\dot{z}}{1\pm z^n}$; hence $\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}=\frac{1}{1\pm z^n}\times$ $\overline{\overline{r+1}\times az^r\dot{z}-naz^r\dot{z}+\frac{naz^r\dot{z}}{1\pm z^n}}+\dot{\mathcal{Q}}=\overline{\overline{r+1}\times a-na}\times$ $\frac{z^r\dot{z}}{1\pm z^n}+\frac{naz^r\dot{z}}{\overline{1\pm z^n}|^2}+\dot{\mathcal{Q}}$; aſſume $na=1$, or $a=\frac{1}{n}$, ſo that the terms $\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}$ and $\frac{naz^r\dot{z}}{\overline{1\pm z^n}|^2}$ may deſtroy each other, and we have $\dot{\mathcal{Q}}=\overline{1-\frac{r+1}{n}}\times\frac{z^r\dot{z}}{1\pm z^n}$; hence if P be the fluent of $\frac{z^r\dot{z}}{1\pm z^n}$, we have $\mathcal{Q}=\overline{1-\frac{r+1}{n}}\times P$; conſequently the fluent required is $\frac{1}{n}\times\frac{z^{r+1}}{1\pm z^n}+\overline{1-\frac{r+1}{n}}$ $\times P$.

154. It frequently happens, that a fluxional equation contains two unknown quantities, in which caſe, they muſt either be ſeparated, or reduced to the fluxion of ſome known fluent; but no general rules can be given for this purpoſe, and the reductions muſt be left to trial and the ſkill of the Analyſt; the following Rules, however, may be of ſome uſe.

RULE

RULE 1.

Multiply or divide the given equation by ſome function of the unknown quantities, ſo as to bring them to a form whoſe fluents may be found by ſome of the rules already given, or to the fluxion of a known fluent.

Examples.

Ex. 1. Let $\frac{\dot x}{x}+\frac{\dot y}{y}=\frac{ax^m\dot x}{y^n}$. Multiply both ſides by nx^ny^n and it becomes $ny^nx^{n-1}\dot x+nx^ny^{n-1}\dot y=nax^{m+n}\dot x$; now the fluent of the firſt part is known from Prop. 7. to be x^ny^n, and the fluent of the other part is found (Art. 37.) to be $\frac{nax^{m+n+1}}{m+n+1}$; hence the equation of the fluents is $x^ny^n=\frac{nax^{m+n+1}}{m+n+1}$.

Ex. 2. Let $\ddot x-x\dot z^2=f\dot z^2$. As $\ddot z$ does not enter into this equation, in deducing the fluxional equation for the fluent, $\dot z$ muſt have been ſuppoſed conſtant. Multiply by $\dot x$ and $\dot x\ddot x-x\dot x\dot z^2=f\dot x\dot z^2$, and as $\dot z$ is conſtant, the fluent is $\frac{1}{2}\dot x^2-\frac{1}{2}x^2\dot z^2=fx\dot z^2$; hence $\dot z=\frac{\dot x}{\sqrt{2fx+x^2}}$, whoſe fluent (Art. 45.) is $z=\text{h. l.}\ \overline{f+x+\sqrt{2fx+x^2}}$.

RULE 2.

Sometimes the fluent may be found, by aſſuming a new variable quantity.

Example.

Let $a\dot z=z\dot x-x\dot x$. Aſſume $z=a+x+v$, then $\dot z=\dot x+\dot v$; hence by ſubſtitution, $a\dot x+a\dot v=a\dot x+x\dot x$
$+v\dot x$

$+v\dot{x}-x\dot{x}$, therefore $a\dot{v}=v\dot{x}$, or $\dot{x}=\frac{a\dot{v}}{v}$; hence (Art. 45.) $x=a\times$ h. l. v; conſequently $z=a+v+a\times$ h. l. v, and by ſubſtituting for v it's value $z-a-x$, we get $x=a\times$ h. l. $\overline{z-a-x}$.

RULE 3.

The fluent may ſometimes be found, by firſt putting the equation into fluxions, making one of the fluxions conſtant.

Example.

Let $\frac{a\dot{x}+y\dot{x}}{\dot{y}}=x+y-\frac{x\dot{y}}{\dot{x}}$. Make $\dot{y}$ conſtant, and put the equation into fluxions, and $\frac{\overline{a+y}\times\ddot{x}}{\dot{y}}+\dot{x}=\dot{x}+\dot{y}+\frac{x\dot{y}\ddot{x}-\dot{x}^2\dot{y}}{\dot{x}^2}$, hence $\frac{\overline{a+y}\times\ddot{x}}{\dot{y}}=\frac{x\dot{y}\ddot{x}}{\dot{x}^2}$, and $\overline{a+y}\times\dot{x}^2=x\dot{y}^2$, conſequently $x^{-\frac{1}{2}}\dot{x}=\overline{a+y}|^{-\frac{1}{2}}\dot{y}$, therefore (Art. 37. and 39.) we have $2x^{\frac{1}{2}}=2\times\overline{a+y}|^{\frac{1}{2}}$.

RULE 4.

If only one of the variable quantities (x or y) enter, ſubſtitute for one fluxion the other multiplied into a new variable quantity.

Example.

Let $y\dot{y}^3\dot{x}=a\dot{x}^4+2a\dot{x}^2\dot{y}^2+a\dot{y}^4$, where x is wanting. Aſſume $z\dot{y}=\dot{x}$, and we get $yz\dot{y}^4=az^4\dot{y}^4+2az^2\dot{y}^4+a\dot{y}^4$,

$+a\dot{y}^4$, or $yz = az^4 + 2az^2 + a$, hence $y = az^3 + 2az + \frac{a}{z}$, therefore $\dot{y} = 3az^2\dot{z} + 2a\dot{z} - \frac{a\dot{z}}{z^2}$, conſequently $z\dot{y}$, or $\dot{x} = 3az^3\dot{z} + 2az\dot{z} - \frac{a\dot{z}}{z}$, whoſe fluent is $x = \frac{3}{4}az^4 + az^2 - a \times$ h. l. z; and if into this equation we ſubſtitute the value of z in terms of y found from the equation $y = az^3 + 2az + \frac{a}{z}$, we ſhall get x in terms of y.

Prop. LXXIX.

In any fluxional equation of the ſecond order, where the fluxion of one of the variable quantities ($\dot{x}$) *is conſtant; if for* $\ddot{y}$ *we ſubſtitute* $\frac{-\dot{y}\ddot{x}}{\dot{x}}$, *the equation will be transformed into one in which* $\dot{y}$ *is conſtant.*

155. For ſuppoſe the value of y to be expreſſed by $a + bx + cx^2 + dx^3 +$ &c. then $\frac{\dot{y}}{\dot{x}} = b + 2cx + 3dx^2 +$ &c. Make $\dot{x}$ conſtant, and take the fluxion, and $\frac{\ddot{y}}{\dot{x}} = 2c\dot{x} + 6dx\dot{x} +$ &c. Now make $\dot{y}$ conſtant, and $\frac{-\dot{y}\ddot{x}}{\dot{x}^2} = 2c\dot{x} + 6dx\dot{x} +$ &c. when therefore $\dot{x}$ is conſtant the value of $\frac{\ddot{y}}{\dot{x}}$ is the ſame as $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$ when $\dot{y}$ is conſtant; hence if in any fluxional equation in which $\dot{x}$ is conſtant, we ſubſtitute for $\frac{\ddot{y}}{\dot{x}}$ the quantity $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$, or for $\ddot{y}$ the quantity $\frac{-\dot{y}\ddot{x}}{\dot{x}}$, we ſhall transform the equation into one in which $\dot{y}$ is conſtant.

By this ſubſtitution, the fluents of fluxional equations may ſometimes be found.

Example.

Let $\dot{x}\dot{y} - x\ddot{y} - a\ddot{y} - \frac{x\dot{y}^2}{b} = 0$, in which $\dot{x}$ is conſtant, as $\ddot{x}$ does not enter. Subſtitute $\frac{-\dot{y}\ddot{x}}{\dot{x}}$ for $\ddot{y}$ (in which caſe $\dot{y}$ becomes conſtant) and we get $\dot{x}\dot{y} + x \times \frac{\dot{y}\ddot{x}}{\dot{x}} + a \times \frac{\dot{y}\ddot{x}}{\dot{x}} - \frac{x\dot{y}^2}{b} = 0$, or $\dot{x}^2 + x\ddot{x} + a\ddot{x} - \frac{x\dot{x}\dot{y}}{b} = 0$, whoſe fluent is $x\dot{x} + a\dot{x} - \frac{x^2\dot{y}}{2b}$, which, as the fluxion is $= 0$, muſt be equal to ſome conſtant quantity; let it be $c\dot{y}$ [*], and then $\dot{y} = \frac{2bx\dot{x}}{2bc + x^2} + \frac{2ab\dot{x}}{2bc + x^2}$, whoſe fluents (Art. 45. and 46.) are $y = b \times L + a \times \sqrt{\frac{2b}{c}} \times A$, where A is a circular arc, whoſe radius is 1 and tangent $\frac{x}{\sqrt{2bc}}$, and $L = \text{h.l.}\,\overline{2bc + x^2}$.

If the fluent cannot be obtained by theſe means, or any other artifices of a like kind, recourſe muſt be had to infinite ſeries, an inſtance of which the Reader may ſee in Art. 111.

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[*] It is eaſy to conceive that this conſtant quantity muſt be ſuch as $c\dot{y}$, becauſe the equation, after taking the fluent this firſt time, aroſe from taking the fluxion of the fluential equation, and therefore $\dot{x}$ or $\dot{y}$ muſt neceſſarily enter into every term.

SECTION XI.

ON THE SUMMATION OF SERIES.

Prop. LXXX.

To find the ſum of $1^n x + 2^n x^2 + 3^n x^3 + \&c. \dots s^n x^s$.

156. Aſſume $x + x^2 + x^3 + \&c. \dots x^s = \frac{x^{s+1} - x}{x - 1} = a$; take the fluxion of both ſides, divide by $\dot{x}$ and multiply by x; repeat this operation and you will raiſe the powers of the natural numbers an unit every time; hence

$$1\,x + 2\,x^2 + 3\,x^3 + \&c. \dots s\,x^s = \frac{x\dot{a}}{\dot{x}} = b;$$

$$1^2 x + 2^2 x^2 + 3^2 x^3 + \&c. \dots s^2 x^s = \frac{x\dot{b}}{\dot{x}} = c;$$

$$1^3 x + 2^3 x^2 + 3^3 x^3 + \&c. \dots s^3 x^s = \frac{x\dot{c}}{\dot{x}} = d;$$

Thus we may continue the operation to any power.

Prop.

PROP. LXXXI.

To find the ſum of $1.2.3x+2.3.4x^2+3.4.5x^3+$ *&c.* . $\overline{s-2}.\overline{s-1}.sx^{s-2}$.

157. Aſſume as before, take the fluxion and divide by $\dot{x}$, repeat this operation till you have gotten the number of factors, and then multiply by x; hence

$$1+2x+3x^2+4x^3+\&c.\ \dots\dots\ sx^{s-1}=\frac{\dot{a}}{\dot{x}}=b;$$

$$1.2+2.3x+3.4x^2+\&c.\ \dots\ \overline{s-1}.sx^{s-2}=\frac{\dot{b}}{\dot{x}}=c;$$

$$1.2.3x+2.3.4x^2+\&c.\overline{s-2}.\overline{s-1}.sx^{s-2}=\frac{x\dot{c}}{\dot{x}}=d.$$

PROP. LXXXII.

Given $ax^n+bx^{2n}+cx^{3n}+$ *&c.* $+mx^{vn}=A$; *to find* $\overline{p+n}\times\overline{q+n}\times ax^n+\overline{p+2n}\times\overline{q+2n}\times bx^{2n}+\&c.\ \dots\ \overline{p+vn}\times\overline{q+vn}.mx^{vn}$.

158. Multiply the given equation by x^p, and $ax^{p+n}+bx^{p+2n}+\&c.=Ax^p=B$; take the fluxion and divide by $\dot{x}$, and $\overline{p+n}\times ax^{p+n-1}+\overline{p+2n}\times bx^{p+2n-1}+\&c.=\frac{\dot{B}}{\dot{x}}$, divide by x^{p-1}, and $\overline{p+n}\times ax^n+\overline{p+2n}\times bx^{2n}+\&c.=\frac{\dot{B}}{x^{p-1}\dot{x}}=C$. Now multiply this equation by x^q, take the fluxion and divide by $x^{q-1}\dot{x}$, and we get $\overline{p+n}\times\overline{q+n}\times ax^n+\overline{p+2n}\times\overline{q+2n}\times bx^{2n}+\&c.=\frac{\dot{\overline{Cx^q}}}{x^{q-1}\dot{x}}$.

In this manner any factors may be introduced, by multiplying by ſuch powers of x as ſhall produce the factors required.

Prop. LXXXIII.

Let the ſum of $\frac{2x}{1} - \frac{4x^3}{3} + \frac{6x^5}{5} -$ *&c. ad infinitum be required.*

159. By Art. 46. $\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} -$ &c. $= A$, A being an arc of a circle whoſe radius $= 1$, tangent $= x$. Multiply by x, and $\frac{x^2}{1} - \frac{x^4}{3} + \frac{x^6}{5} -$ &c. $= Ax$, hence $\frac{2x}{1} - \frac{4x^3}{3} + \frac{6x^5}{5} -$ &c. $= \frac{A\dot{x} + x\dot{A}}{\dot{x}} =$ (becauſe $\dot{A} = \frac{\dot{x}}{1+x^2}$ by Art. 46.) $A + \frac{x}{1+x^2}$.

If $x = 1$, then $\frac{2}{1} - \frac{4}{3} + \frac{6}{5} -$ &c. $= A + \frac{1}{2}$.

Prop. LXXXIV.

To ſum ſeries by means of the fluent of $vx^n\dot{x}$, v *being* $=$ *h. l.* $\frac{1}{1-x}$.

160. By Art. 149. the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times \overline{\frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \text{\&c.}} - v = v \times \overline{\frac{x^{n+1}}{n+1} - \frac{1}{n+1}} + \frac{x^{n+1}}{\overline{n+1} \times \overline{n+1}} + \frac{x^n}{\overline{n+1} \times n} + \frac{x^{n-1}}{\overline{n+1}.\overline{n-1}} +$ &c. But $v =$ hyp. log. $\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. ad inf.

hence $v x^n \dot{x} = x^{n+1}\dot{x} + \frac{1}{2} x^{n+2}\dot{x} + \frac{1}{3} x^{n+3}\dot{x} + \frac{1}{4} x^{n+4}\dot{x} +$ &c. whoſe fluent is $\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.\overline{n+3}} + \frac{x^{n+4}}{3.\overline{n+4}} + \frac{x^{n+5}}{4.\overline{n+5}}$ + &c. Make theſe two fluents equal, and we have $\frac{v}{n+1} \times \overline{x^{n+1}-1} + \frac{x^{n+1}}{\overline{n+1} \times \overline{n+1}} + \frac{x^n}{\overline{n+1} \times n} + \frac{x^{n-1}}{\overline{n+1} \times \overline{n-1}}$ + &c. to $n+1$ terms $= \frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.\overline{n+3}} + \frac{x^{n+4}}{3.\overline{n+4}}$ + &c. ad infinitum.

161. If $n=0$, then $\frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4}$ + &c. ad inf. $= v \times \overline{x-1} + x$; hence if $x=1$, $\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4}$ + &c. $= 1$.

162. Since $\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} +$ &c. $= vx - v + x$, multiply by $\dot{x}$ and $\frac{x^2\dot{x}}{1.2} + \frac{x^3\dot{x}}{2.3} + \frac{x^4\dot{x}}{3.4}$ + &c. $= vx\dot{x} - v\dot{x} + x\dot{x}$; now by Art. 160. the fluent of $vx\dot{x}$ is $\frac{1}{2} vx^2 - \frac{1}{2} v + \frac{1}{4} x^2 + \frac{1}{2} x$; alſo the fluent of $v\dot{x}$ is $vx - v + x$; hence the fluent of $vx\dot{x} - v\dot{x} + x\dot{x}$ is $\frac{1}{2} vx^2 - vx + \frac{1}{2} v - \frac{1}{2} x + \frac{3}{4} x^2$; conſequently (*B*) $\frac{x^3}{1.2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{3.4.5}$ + &c. $= \frac{1}{2} vx^2 - vx + \frac{1}{2} v - \frac{1}{2} x + \frac{3}{4} x^2$. Aſſume $\frac{1}{2} vx^2 - vx + \frac{1}{2} v = 0$, or $x^2 - 2x + 1 = 0$, hence $x = 1$; make $x = 1$, and $\frac{1}{1.2.3} + \frac{1}{2.3.4}$ + &c. $= \frac{1}{4}$.

Let $x = \frac{1}{4}$, then $v =$ h. l. $\frac{4}{3}$; hence $\frac{1}{2.3.4} \times \frac{1}{4^3} + \frac{1}{2.3.4} \times \frac{1}{4^4}$ + &c. $= \frac{9}{32} \times$ h. l. $\frac{4}{3} - \frac{5}{64}$.

Let $x=\frac{1}{2}$, then $v=$ h. l. 2; hence $\frac{1}{1.2.3}\times\frac{1}{8}+\frac{1}{3.4.5}\times\frac{1}{16}+\&c.=\frac{1}{8}\times$ h. l. $2-\frac{1}{16}$. Thus by aſſuming x and determining v from it, we may find the ſum of the correſponding ſeries.

In like manner by multiplying B by $\dot{x}$ and taking the fluent, we ſhall get four factors in the denominator, 1.2.3.4, 2.3.4.5, &c. or if we multiply by $x\dot{x}$ and take the fluent we ſhall get the factors 1.2.3.5, 2.3.4.6, &c. And, in like manner, we may add what factors we pleaſe, by multiplying by ſuch a power of x as will produce that factor. If the Reader wiſh to ſee more inſtances, he may conſult A. de MOIVRE's *Miſcel. Anal.* Lib. VI.

PROP. LXXXV.

To ſum ſeries from the fluent of $vx^n\dot{x}$, where v is a circular arc, whoſe radius is unity and tangent x.

163. By Art. 150, the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x^{n-1}}{n-1}-\&c.\mp v}$, where the ſign of v is $+$ or $-$ according as $\frac{n+1}{2}$ is odd or even, when n is an *odd* number. By Art. 46. $v=x-\frac{x^3}{3}+\frac{x^5}{5}-\&c.$ hence $vx^n\dot{x}=x^{n+1}\dot{x}-\frac{x^{n+3}\dot{x}}{3}+\frac{x^{n+5}\dot{x}}{5}-\&c.$ whoſe fluent is $\frac{x^{n+2}}{n+2}-\frac{x^{n+4}}{3.\overline{n+4}}+\frac{x^{n+6}}{5.\overline{n+6}}-\&c.$ Make theſe fluents

equal, and we have $\frac{1}{n+1} \times vx^{n+1} \mp \overline{v - \frac{x^n}{n} + \frac{x^{n-1}}{n-1} -}$ &c. $= \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3 \cdot \overline{n+4}} + \frac{x^{n+6}}{5 \cdot \overline{n+6}} -$ &c. ad. inf.

Let $\frac{n+1}{2}$ be an *even* number, and aſſume $\overline{vx^{n+1} -}$ $v = 0$, and then $x = 1$; hence $\frac{1}{n+1} \times \mp \frac{1}{n} + \frac{1}{n-2} -$ &c. to $\frac{n+1}{2}$ terms, is equal to $\frac{1}{n+2} - \frac{1}{3 \cdot \overline{n+4}} + \frac{1}{5 \cdot \overline{n+6}}$ − &c. ad infinitum.

If $n = 3$, then $\frac{1}{1 \cdot 5} - \frac{1}{3 \cdot 7} + \frac{1}{5 \cdot 9} -$ &c. ad infinitum $= \frac{1}{4} \times \overline{-\frac{1}{3} + 1} = \frac{1}{6}$.

Let $\frac{n+1}{2}$ be an *odd* number, and aſſume $n = 1$, $x = 1$; then v becomes an arc of 45°; and we get $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} -$ &c. ad infinitum = arc $45° - \frac{1}{2}$.

If n be an *even* number, then by Art. 150. we get in like manner,

$$\frac{1}{n+1} \times \overline{vx^{n+1} - \frac{x^n}{n} + \frac{x^{n-2}}{n-2} - \text{\&c.} \mp \text{h. l.} \sqrt{1+x^2}} = \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3 \cdot \overline{n+4}} + \frac{x^{n+6}}{5 \cdot \overline{n+6}} - \text{\&c. ad infinitum,}$$

where the number of terms to be taken in the firſt ſeries is $\frac{1}{2} n$, the firſt and laſt terms excepted, and the ſign of the laſt term is + or −, according as $\frac{1}{2} n$ is odd or even.

If $n=2$, and $x=1$, then v becomes an arc of 45°; and we get $\frac{1}{1.4}-\frac{1}{3.6}+\frac{1}{5.8}-$ &c. ad infinitum $=\overline{\frac{1}{3}\times\text{arc }45^\circ-\frac{1}{2}+\text{h. l. }}\sqrt{2}$. For more upon this ſubject, ſee A. de MOIVRE's *Miſcel. Anal.* Lib. VI.

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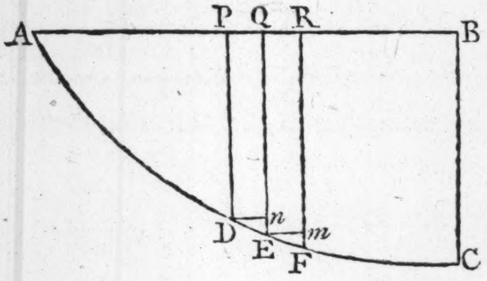
SECTION XII.

ON THE MAXIMA AND MINIMA OF CURVES.

PROP. LXXXVI.

TO find the nature of curves, in which ſome conditions remaining invariable, others are the greateſt or leaſt poſſible.

164. Let *ABC* be any curvilinear area, *PD*, *RF* two fixed ordinates indefinitely near to each other, and the ordinate *QE* an arithmetic mean between them, ſo that $En = Fm$, *Dn*, *Em* being parallel to *AB*. Now it is manifeſt that the nature of the curve *DEF* muſt depend upon the poſition of the point *E*, as by vary-



varying the poſition of that point you muſt neceſſarily vary the curve; upon the ſituation therefore of this intermediate ordinate, the determination of the equation to the curve, from the data, will depend. Hence PQ, QR, are the only variable quantities.

165. Let any given quantity M be made up of A, B, C, D, E &c. or let $A+B+C+D+E+\&c.=M$, and at the ſame time let ſome other quantity m be required to be a maximum or minimum, and let the correſponding parts of m be a, b, c, d, e, &c. and conſequently $a+b+c+d+e+\&c.=m$, M and m being expreſſed in terms of the ſame variable quantities. Now let us ſuppoſe all the quantities in each to remain conſtant, except two which correſpond, that is, let C and D, c and d be alone variable; then $C+D$ is conſtant, and to ſatisfy the other condition, $c+d$ muſt be a maximum or minimum; hence (Art. 21.) $\dot{C}+\dot{D}=0$, $\dot{c}+\dot{d}=0$, from which two equations we may get the relation of the variable quantities which compoſe them, and which will be found ſufficient to determine the nature of the curve.

PROP. LXXXVII.

Given the points A and C, to find the curve in which a body will deſcend from A to C in the leaſt time poſſible.

166. Put $PD=m$, $QE=n$, $En=Fm=a$, the conſtant quantities, $v=PQ=Dn$, $w=QR=Em$; then $DE=\sqrt{a^2+v^2}$ and $EF=\sqrt{a^2+w^2}$. Now AB being parallel to the horizon, the velocities at D and E are as $\sqrt{m}$ and $\sqrt{n}$ by Mechanics; alſo the times be-

ing as the ſpaces directly and velocities inverſely, the times through *DE*, *EF* will be as $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$; hence as *AB* is given, v, w are two parts of this given quantity, whoſe ſum $v+w$ is conſtant; alſo $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$ are the two correſponding parts of the minimum, whoſe ſum $\frac{\sqrt{a^2+v^2}}{\sqrt{m}} + \frac{\sqrt{a^2+w^2}}{\sqrt{n}} =$ minimum (Art. 165.); hence $\dot{v}+\dot{w}=0$, and $\frac{v\dot{v}}{\sqrt{m}\times\sqrt{a^2+v^2}} + \frac{w\dot{w}}{\sqrt{n}\times\sqrt{a^2+w^2}} = 0$; $\therefore$ $\dot{w} = -\dot{v}$, conſequently $\frac{v\dot{v}}{\sqrt{m}\times\sqrt{a^2+v^2}} - \frac{w\dot{v}}{\sqrt{n}\times\sqrt{a^2+w^2}} = 0$, and $\frac{v}{\sqrt{m}\times\sqrt{a^2+v^2}} = \frac{w}{\sqrt{n}\times\sqrt{a^2+w^2}}$; now theſe are two ſimilar quantities, which expreſs (in their ultimate ſtate) the fluxion of the abſciſſa divided by the ſquare root of the ordinate × fluxion of the curve; two ſucceſſive values of this quantity therefore being equal to each other ſhows the quantity itſelf to be conſtant; hence put $AP=x$, $PD=y$, $AD=z$, and we have $\frac{\dot{x}}{\sqrt{y}\times\dot{z}} = \frac{1}{\sqrt{r}}$ a conſtant quantity, which is the property of a cycloid, the diameter of whoſe generating ſemicircle is r.

PROP.

PROP. LXXXVIII.

To determine the nature of the curve AC whoſe length is given, and it's area a maximum.

167. The ſame notation remaining, we have $DE + EF = \sqrt{a^2+v^2} + \sqrt{a^2+w^2}$ a conſtant quantity, the ſum of two parts of the given curve line AC; alſo $mv+nw$ is the ſum of the two correſponding parts of the maximum; hence (Art. 165.) $mv+nw=$ max. $\therefore m\dot{v}+n\dot{w}=0$, and $\frac{v\dot{v}}{\sqrt{a^2+v^2}} + \frac{w\dot{w}}{\sqrt{a^2+w^2}} = 0$; hence $\frac{m\dot{v}}{n} = -\dot{w}$, therefore $\frac{v\dot{v}}{\sqrt{a^2+v^2}} - \frac{mw\dot{v}}{n\sqrt{a^2+v^2}} = 0$, conſequently $\frac{v}{m\sqrt{a^2+v^2}} = \frac{w}{n\sqrt{a^2+w^2}}$; which being ſimilar quantities, we have $\frac{\dot{x}}{y\dot{z}} = \frac{1}{r}$ a conſtant quantity, or $r\dot{x} = y\dot{z}$ the equation to a circle by Art. 46.

PROP. LXXXIX.

Let the ſurface of the ſolid generated by the revolution of the curve AC about AB be given; to find the nature of the curve when the ſolid is a maximum.

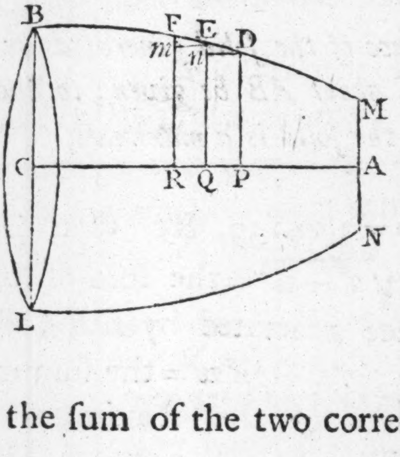
168. Put $p=3,14159$, &c. then (Art. 55.) $2pm\sqrt{a^2+v^2} + 2pn\sqrt{a^2+w^2} =$ the ſum of the two parts of the given ſurface generated by $DE+EF$, a conſtant quantity; alſo $pm^2v+pn^2w=$ the ſum of the two correſponding parts of the maximum generated by $PQED$, $QRFE$; hence $pm^2v + pn^2w =$ max. $\therefore$ (neglecting the conſtant multiplier p) $m^2\dot{v}+n^2\dot{w}=0$,

and $\frac{mv\dot{v}}{\sqrt{a^2+v^2}} + \frac{nw\dot{w}}{\sqrt{a^2+w^2}} = 0$; hence $\dot{w} = -\frac{m^2\dot{v}}{n^2}$, which ſubſtituted for $\dot{w}$ in the ſecond equation, we get $\frac{v}{m\sqrt{a^2+v^2}} = \frac{w}{\sqrt{a^2+w^2}}$, which are the ſame quantities as in the laſt caſe; hence the curve is a circle.

PROP. XC.

To find the nature of the curve which generates a ſolid of the leaſt reſiſtance, when moving in a fluid in the direc- of it's axis, it's greateſt diameter BL and length AC being given.

169. By the Principles of Hydroſtatics, the reſiſtance againſt DE is as $\frac{ma^3}{a^2+v^2}$, and againſt EF as $\frac{na^3}{a^2+w^2}$; hence the ſum of the two parts of the quantity which is to be a minimum $= \frac{ma^3}{a^2+v^2} + \frac{na^3}{a^2+w^2}$; alſo as AC is



given, $v+w$, the ſum of the two correſponding parts of the given quantity, is conſtant; therefore $-\frac{2ma^3v\dot{v}}{\overline{a^2+v^2}^2}$

$-2na^3w\dot{w}$

$-\frac{2na^3w\dot{w}}{\overline{a^2+w^2}^2} = 0$, and $\dot{v} + \dot{w} = 0$; hence $\dot{v} = -\dot{w}$; conſequently, by ſubſtitution, $\frac{ma^3v}{\overline{a^2+v^2}^2} = \frac{na^3w}{\overline{a^2+w^2}^2}$, which being ſimilar quantities, we have $\frac{y\dot{y}^3\dot{x}}{\dot{z}^4} = r$ a given quantity, which is the fluxional equation of the curve. That the curve does not meet the axis at *A*, appears from hence; $y = r \times \frac{\dot{z}^4}{\dot{y}^3\dot{x}} = r \times \frac{ED^4}{En^3 \times Dn}$, where the numerator muſt evidently be greater than the denominator, and therefore *y* muſt be greater than *r*.

If the greateſt diameter *BL*, and *area BMNL* be given, then $mv+nw$ will be given, conſequently $m\dot{v} + n\dot{w} = 0$, which gives $\frac{\dot{y}^3\dot{x}}{\dot{z}^4} = r$ for the equation of the curve.

If the greateſt diameter and *bulk* be given, then inſtead of $v+w$ being given, pm^2v+pn^2w will be given (Art. 165.); hence $m^2\dot{v}+n^2\dot{w}=0$, which gives $\frac{\dot{y}^3\dot{x}}{y\dot{x}^4} = r$, for the equation of the curve.

Although *PDEQ*, *QEFR* are in theſe figures taken as increments, yet we reaſon upon them as fluxions, conceiving their limiting ratio to be taken, and conſequently the concluſions are mathematically true.

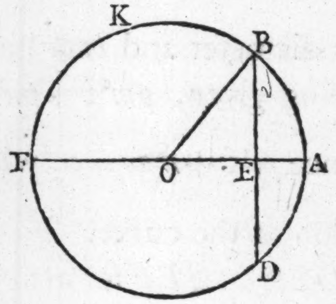
SECTION XIII.

MISCELLANEOUS PROPOSITIONS.

Prop. XCI.

GIVEN the ſine EB of an arc AB of a circle; to find the ſine of n times AB.

170. Let $AB=z$, and $AK=nz$; put $OB=1$, $y=OE$ the coſine of AB, $v=$ the ſine $BE=\sqrt{1^2-y^2}$, $x=$ the coſine of AK, then $\sqrt{1^2-x^2}=$ the ſine of



AK. Now (Art. 46.) $\dot{z} : -\dot{y} :: 1 : \sqrt{1^2-y^2}$, $\therefore \dot{z} = \frac{-\dot{y}}{\sqrt{1^2-y^2}}$; for the ſame reaſon the fluxion of nz, or $n\dot{z} = \frac{-\dot{x}}{\sqrt{1^2-x^2}}$; hence $\frac{\dot{x}}{\sqrt{1^2-x^2}} = \frac{n\dot{y}}{\sqrt{1^2-y^2}}$; multiply both denominators by $\sqrt{-1}$, and $\frac{\dot{x}}{\sqrt{x^2-1^2}} = \frac{n\dot{y}}{\sqrt{y^2-1^2}}$, whoſe fluent (Art 45.) is h. l. $\overline{x+\sqrt{x^2-1^2}} = n \times$ h. l. $\overline{y+\sqrt{y^2-1^2}}$; hence (Art. 109.) $x + \sqrt{x^2-1^2} = \overline{y+\sqrt{y^2-1^2}}|^n =$ (Art. 33.) $y^n + ny^{n-1}\sqrt{y^2-1} + n.\frac{n-1}{2}.y^{n-2} \times \overline{y^2-1} + n.\frac{n-1}{2}.\frac{n-2}{3}.y^{n-3} \times \sqrt{y^2-1} \times \overline{y^2-1} +$ &c. Now as this equation conſiſts of quantities, partly poſſible and partly impoſſible, $\sqrt{x^2-1}$ and $\sqrt{y^2-1}$ being impoſſible, it is manifeſt that the poſſible and impoſſible parts muſt be reſpectively equal. Hence aſſuming the impoſſible parts equal, we have, $\sqrt{x^2-1} = ny^{n-1}\sqrt{y^2-1} + n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3} \times \sqrt{y^2-1} \times \overline{y^2-1} +$ &c. Multiply both ſides by $\sqrt{-1}$, and $\sqrt{1-x^2} = ny^{n-1}\sqrt{1-y^2} + n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3} \times \sqrt{1-y^2} \times \overline{y^2-1} +$ &c. $=$ (becauſe $v = \sqrt{1-y^2}$, and $-v^2 = y^2 - 1$) $ny^{n-1}v - n.\frac{n-1}{2}.\frac{n-2}{3}.y^{n-3}v^3 +$.&c. the ſine of *AK*.

PROP. XCII.

Given as before, to find the coſine of AK.

171. Aſſume the poſſible parts of the above equation equal, and we have $x = y^n + n.\frac{n-1}{2}y^{n-2} \times \overline{y^2-1}$ + &

$+ \&c. = y^n - n.\frac{n-1}{2}\,y^{n-2}v^2 + \&c.$ the coſine of *AK*.

Prop. XCIII.

Given as before, to find the tangent of AK.

172. Put $t =$ tangent of *AB*, then by plain trig. $t = \frac{v}{y}$, radius being unity; hence the tangent of $AK =$

$$\frac{\text{ſin. } AK}{\text{coſ. } AK} = \frac{n\,y^{n-1}v - n.\frac{n-1}{2}.\frac{n-2}{3}.y^{n-3}v^3 + \&c.}{y^n - n.\frac{n-1}{2}y^{n-2}v^2 + \&c.} = (\text{by}$$

dividing the numerator and denominater by y^n)

$$\frac{\frac{nv}{y} - n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{v^3}{y^3} + \&c.}{1 - n.\frac{n-1}{2}.\frac{v^2}{y^2} - \&c.} = \frac{nt - n.\frac{n-1}{2}.\frac{n-2}{3}.t^3 + \&c.}{1 - n.\frac{n-1}{2}.t^2 + \&c.}.$$

Prop. XCIV.

To reſolve $v^{2n} - 2xv^n + 1 = 0$ into it's quadratic diviſors, x being equal to or leſs than unity.

173. Retaining the notation in Art. 170. we have $x + \sqrt{x^2 - 1^2} = \overline{y + \sqrt{y^2 - 1^2}}\,\rvert^n$. Put $v = y + \sqrt{y^2 - 1^2}$; tranſpoſe y and ſquare both ſides, and we get $v^2 - 2yv = -1^2$, $\therefore v^2 - 2yv + 1^2 = 0$. Alſo $v^n = x + \sqrt{x^2 - 1^2}$, hence by tranſpoſing x and proceding as before, we get $v^{2n} - 2xv^n + 1 = 0$, the given equation, of which we have one quadratic diviſor $v^2 - 2yv + 1^2 = 0$, v being the ſame in both equations. Now if to the arc *AK*, we add 360°, 2 × 360°, &c. we ſhall come again

to the ſame point K, and conſequently we ſhall have the ſame coſine, or x; hence x is the coſine of AK, $360° + AK$, $2 \times 360° + AK$, &c. But y is the coſine of an n^{th} part of that arc whoſe coſine is x; hence y is the coſine of $\frac{AK}{n}$, $\frac{360° + AK}{n}$, $\frac{2 \times 360° + AK}{n}$, &c. which coſines call a, b, c, &c. ſubſtitute therefore theſe values for y in the equation $v^2 - 2yv + 1^2 = 0$, and we get $v^2 - 2av + 1^2 = 0$, $v^2 - 2bv + 1^2 = 0$, $v^2 - 2cv + 1^2 = 0$, &c. for the quadratic diviſors required; hence $\overline{v^2 - 2av + 1^2} \times \overline{v^2 - 2bv + 1^2} \times$ &c. $= v^{2n} - 2xv^n + 1^{2n}$, retaining the power of the radius in the laſt term. Although there are an infinite number of arcs whoſe coſines are x, and conſequently an infinite number of correſponding values of y, yet there are only n *different* values of y, becauſe after taking n arcs, $\frac{AK}{n}$, $\frac{360 + AK}{n}$, &c. the coſines will return again the ſame.

174. If $x = \pm 1$, or if AK be taken equal to the whole circumference, or half the circumference, the equation becomes $v^{2n} \mp 2v^n + 1 = 0$, whoſe ſquare root is $v^n \mp 1 = 0$; now as every equation which is a ſquare muſt have to every root another equal to it, the equation $v^n \mp 1 = 0$ muſt contain the ſame roots as $v^{2n} \mp 2v^n + 1 = 0$; the roots therefore of $v^n \mp 1 = 0$ are found in like manner.

175. Hence we may find the quadratic diviſors of $v^{2n} - 2xr^n v^n + r^{2n} = 0$, which is the equation $v^{2n} - 2xv^n + 1 = 0$, having it's roots multiplied by r (Alg. Art. 282.); multiplying the roots therefore of the above

quadratics by r, we have $v^2-2arv+r^2=0$, $v^2-2brv+r^2=0$, &c. for the quadratics required. If $AK=90°$, then $x=0$, and the equation becomes $v^{2n}+r^{2n}=0$.

Prop. XCV.

To reſolve $\frac{1}{1-2xv^n+v^{2n}}$ *into* $\frac{P-Qv}{1-2av+v^2}+\frac{R-Sv}{1-2bv+v^2}+$ &c.

176. Let the roots of $1-2xv^n+v^{2n}=0$, be $\frac{1}{m}$, $\frac{1}{p}$, $\frac{1}{q}$, &c. then as this is a recurring equation (Alg. Art. 289) the correſponding roots will be $m,p,q,$&c. Aſſume $\frac{1}{1-2xv^n+v^{2n}}=\frac{A}{1-mv}+\frac{B}{1-pv}+\frac{C}{1-qv}+$ &c. then reducing theſe to a common denominator, we have $A\times\overline{1-pv}\times\overline{1-qv}\times$&c.$+B\times\overline{1-mv}\times\overline{1-qv}\times$&c.$+$&c.$=1$; let $1-mv=0$, then $v=\frac{1}{m}$, hence $=$ $A\times\overline{1-\frac{p}{m}}\times\overline{1-\frac{q}{m}}\times$&c.$=1$, or $A\times\frac{\overline{m-p}}{m}\times\frac{\overline{m-q}}{m}\times$ &c. $=1$, or if $w=\overline{m-p}\times\overline{m-q}\times$ &c. $A\times\frac{w}{m^{2n-1}}=1$, hence $A=\frac{m^{2n-1}}{w}$. In like manner we find B, C, &c. by making $1-pv=0$, $1-qv=0$, &c. Now as $1-2xv^n+v^{2n}=\overline{v-m}\times\overline{v-p}\times\overline{v-q}\times$&c. take the fluxion, omitting $\dot{v}$, and $2nv^{2n-1}-2nxv^{n-1}=\overline{v-p}\times\overline{v-q}\times$&c.$+\overline{v-m}\times\overline{v-q}\times$&c.$+$&c. now let $v=m$, and it becomes $2nm^{2n-1}-2nxm^{n-1}=\overline{m-p}\times\overline{m-q}$

$\overline{m-q}\times$&c. $=w$, hence $A\left(=\frac{m^{2n-1}}{w}\right)=\frac{m^{2n-1}}{2nm^{2n-1}-2nxm^{n-1}}$ $=\frac{m^n}{2nm^n-2nx}$. For the ſame reaſon $B=\frac{p^n}{2np^n-2nx}$, &c. Now $\frac{A}{1-mv}+\frac{B}{1-pv}=\frac{A+B-\overline{pA+mB}\times v}{1-2av+v^2}$; then as $1-2av+v^2=\overline{1-mv}\times\overline{1-pv}=1-\overline{m+p}\times v$ $+mpv^2$, we have $m+p=2a$, and $mp=1$. Alſo $A=$ $\frac{m^n}{2nm^n-2nx}$, $B=\frac{p^n}{2np^n-2nx}$; hence $A+B=$ $\frac{4nm^np^n-2xn\times\overline{m^n+p^n}}{4n^2m^np^n-4n^2x\times\overline{m^n+p^n}+4n^2x^2}$. But as $v^{2n}-2xv^n+1$ $=0$, $\therefore\ v^n+\frac{1}{v^n}=2x$; now for v ſubſtitute m, and $m^n+\frac{1}{m^n}=2x$; but $mp=1$, and $p=\frac{1}{m}$; hence $m^n+p^n=$ $2x$; conſequently $A+B=\frac{4n-4nx^2}{4n^2-8n^2x^2+4n^2x^2}=$ $\frac{4n\times\overline{1-x^2}}{4n^2\times\overline{1-x^2}}=\frac{1}{n}$. Alſo $pA+mB=\frac{pm^n}{2nm^n-2nx}+$ $\frac{mp^n}{2np^n-2nx}=$ $\frac{2nm\times m^np^n+2np\times m^np^n-2nxp\times m^n-2nxm\times p^n}{4n^2\times\overline{1-x^2}}$ (the common denominator being the ſame as in the value of $A+B$) $=$ (as $pm=1$, $m+p=2a$) $\frac{2n\times 2a-2nxpm\times m^{n-1}-2nxpm\times p^{n-1}}{4n^2\times\overline{1-x^2}}=$ $\frac{4na-2nx\times\overline{m^{n-1}+p^{n-1}}}{4n^2\times\overline{1-x^2}}$. Now $m^n+p^n=2x$, where x is the coſine of an arc which is to the arc whoſe coſine is a as $n : 1$; for the ſame reaſon $m^{n-1} + p^{n-1} = 2e$, if e be the coſine of an arc which is to the arc whoſe coſine is a as $n-1 : 1$; therefore $pA + mB = \frac{4na - 2nx \times 2e}{4n^2 \times \overline{1 - x^2}} = \frac{a - ex}{n - nx^2}$. Hence $\frac{A}{1 - mv} + \frac{B}{1 - pv} = \frac{\frac{1}{n} - \frac{a - ex}{n - nx^2} \times v}{1 - 2av + v^2}$. Conſequently $\frac{1}{1 - 2xv^n + v^{2n}}$

$$= \frac{\frac{1}{n} + \frac{a - ex}{n - nx^2} \times v}{1 - 2av + v^2} + \frac{\frac{1}{n} + \frac{b - fx}{n - nx^2} \times v}{1 - 2bv + v^2} + \&c.$$

where f is found from b in the ſame manner that e is found from a; and ſo on.

177. If x be negative, the given quantity becomes $\frac{1}{1 + 2xv^n + v^{2n}}$.

178. In like manner, $\frac{1}{1 \pm v^n}$ will be found equal to $\frac{A}{1 - mv} + \frac{B}{1 - pv} + \&c.$ where $A = \frac{1}{n}$, $B = \frac{1}{n}$, &c. and if $\overline{1 - mv} \times \overline{1 - pv} = 1 - 2av + v^2$, then $\frac{A}{1 - mv} + \frac{B}{1 - pv}$

$$= \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1 - 2av + v^2};$$ and ſo on.

If n be an *odd* number, then one root of $1 + v^n = 0$ is -1; hence $1 + v = 0$ is one of the ſimple equations; hence as the other part is made up of quadratics, we have $\frac{1}{1 + v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1 - 2av + v^2} + \&c. + \frac{\frac{1}{n}}{1 + v}$.

If n be an *odd* number, the equation $1-v^n=0$ contains one ſimple equation, and $\frac{n-1}{2}$ quadratics. Now the equation $1-v^n=0$, has one root $=1$, and, conſequently the ſimple equation is $1-v=0$. Hence $\frac{1}{1-v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1-2av+v^2} + \&c. + \frac{\frac{1}{n}}{1-v}$.

If n be an *even* number, $1-v^n=0$ has two roots -1, $+1$; therefore two of the ſimple equations will be $1-v=0$, $1+v=0$; hence $\frac{1}{1-v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1-2av+v^2}$ $+ \&c. + \frac{\frac{1}{n}}{1-v} + \frac{\frac{1}{n}}{1+v}$.

Prop. XCVI.

Let $\dot{F} = \frac{\dot{v}}{1-2xv^n+v^{2n}}$, *to find* F, x *being conſtant, and the ſame as in the laſt propoſition.*

179. Retaining every thing as in Art. 176. we have

$$\dot{F} = \frac{\frac{1}{n}\dot{v} - \frac{a-ex}{n-nx^2}v\dot{v}}{1-2av+v^2} + \frac{\frac{1}{n}\dot{v} - \frac{b-fx}{n-nx^2}v\dot{v}}{1-2bv+v^2} + \&c.$$

the fluent of each of which quantities is found as in Art. 137.

Prop. XCVII.

Let $\dot{F} = \frac{\dot{v}}{1+v^n}$, *n being an* even *number, to find* F.

180. By Art. 178. $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2} + \frac{\frac{2}{n}\dot{v} - \frac{2b}{n}vv}{1 - 2bv + v^2}$ + &c. whoſe fluents are found by Art. 137.

If *n* be an *odd* number, then $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 + v}$, whoſe fluents are found by Art. 137. and 45.

Prop. XCVIII.

Let $\dot{F} = \frac{\dot{v}}{1 - v^n}$, *n being an* even *number, to find F.*

181. By Art. 178. $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 - v}$ + $\frac{\frac{1}{n}\dot{v}}{1 + v}$.

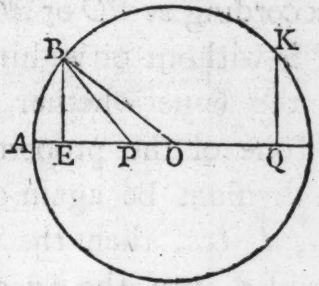
If *n* be an *odd* number, we have $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 + v}$, whoſe fluents are found by Art.. 137. and 45.

Prop. XCIX.

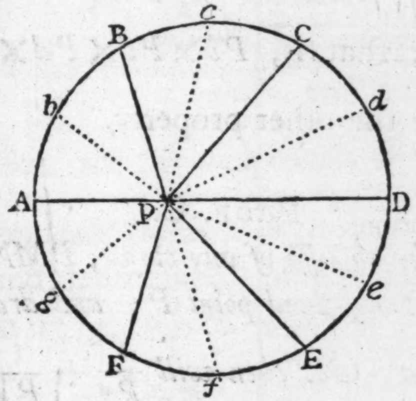
To demonſtrate Cotes*'s properties of the circle.*

182. Retaining every thing as in Art. 173. we have $v^{2n} - 2xv^n + 1^{2n} = 0$, of which $v^2 - 2yv + 1^2$ is a quadratic diviſor. Aſſume any point *P*, and draw *PB*,

PB, and put $v = PO$; then $BO^2 = BP^2 + PO^2 + 2PO \times PE$; that is, $1^2 = BP^2 + v^2 + 2v \times \overline{y - v} = BP^2 - v^2 + 2yv$, hence $BP^2 = v^2 - 2yv + 1^2$. Alſo y is the coſine of $\frac{AK}{n}$, $\frac{360° + AK}{n}$, $\frac{2 \times 360° + AK}{n}$, &c. whoſe coſines are a, b, c, &c. and $\overline{v^2 - 2av + 1^2} \times \overline{v^2 - 2bv + 1^2}$



$\times$&c. $= v^{2n} - 2xv^n + 1^{2n}$. Now let AK be the whole circumference C, then the above arcs are $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. or



the $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, &c. parts of C; that is, if the whole circumference C be divided from A into $n =$ parts at B,

C, D, &c. then theſe arcs are AB, AC, AD, &c. whoſe coſines are a, b, c, &c. hence, from what we have already proved, $PB^2=v^2-2av+1^2$, $PC=v^2-2bv+1^2$, $PD^2=v^2-2cv+1^2$, &c. conſequently $PB^2\times PC^2\times PD^2\times$ &c. $=v^{2n}-2v^n+1^{2n}$, x being equal to 1; hence, by taking the ſquare root, we get $PB\times PC\times PD\times$ &c. $=v^n-1^n$ or $1^n-v^n=PO^n-AO^n$ or AO^n-PO^n,* according as PO or AO is the greateſt, or according as P is without or within the circle, for every thing holds the ſame whether P be within or without. This is one of the properties of the circle.

183. Let theſe diviſions be again divided into two equal parts at b, c, d, &c. then the whole circumference will be divided into the $2n$ equal parts, and therefore from what is already proved, $Pb\times PB\times Pc\times PC\times Pd\times PD\times$ &c. $=AO^{2n}-PO^{2n}$, taking P within, for inſtance; divide this by the above equation, and we get, $\dfrac{Pb\times PB\times Pc\times PC\times Pd\times PD\times \&c.}{PB\times PC\times PD\times \&c.}=\dfrac{AO^{2n}-PO^{2n}}{AO^n-PO^n}$; that is, $Pb\times Pc\times Pd\times$ &c. $=AO^n+PO^n$, which is the other property.

Prop. C.

Let AP be the abſciſſa of any curve, $PMNQ$ an ordinate revolving about any fixed point P; and draw the tangents My, Nx, Qw, &c. then will $\frac{1}{Py}+\frac{1}{Px}+\frac{1}{Pw}+$&c. the ſum of the reciprocal ſubtangents, be a conſtant quantity.

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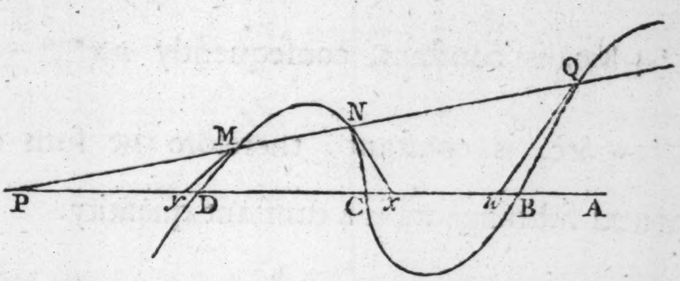
* The letter O is wanting in the center.

184. Let the equation of the curve be $y^n - \overline{a+b}x \times y^{n-1} + \&c. + px^n - qx^{n-1} + \&c. = 0$; and correſponding to *AP* the abſciſſa (x) let a, b, c, &c. be the values of y; then by the Elements of Algebra, Art. 267. $a \times b \times c \times \&c. = px^n - qx^{n-1} + \&c.$ take the fluxion



of each ſide, and $\dot{a}bc\,\&c. + \dot{b}ac\,\&c. + \dot{c}ab\,\&c. + \&c. = npx^{n-1}\dot{x} - \overline{n-1} \times qx^{n-2}\dot{x} + \&c.$ divide this latter equation by the former, and we have $\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} + \&c. = \frac{npx^{n-1}\dot{x} - \overline{n-1} \times qx^{n-2}\dot{x} + \&c.}{px^n - qx^{n-1} + \&c.}$; hence $\frac{\dot{a}}{a\dot{x}} + \frac{\dot{b}}{b\dot{x}} + \frac{\dot{c}}{c\dot{x}} + \&c. = \frac{npx^{n-1} - \overline{n-1} \times qx^{n-2} + \&c.}{px^n - qx^{n-1} + \&c.}$; but $\frac{\dot{a}}{a\dot{x}}$, $\frac{\dot{b}}{b\dot{x}}$, $\frac{\dot{c}}{c\dot{x}}$, &c. are the reciprocals of the ſubtangents *Py*, *Px*, *Pw*, &c. by Art. 23. hence (dividing the numerator and denominator on the right hand ſide of the equation by p, which will not alter it's value)

$$\frac{nx^{n-1} - \overline{n-1} \times \frac{q}{p}x^{n-2} + \&c.}{x^n - \frac{q}{p}x^{n-1} + \&c.} = \frac{1}{Py} + \frac{1}{Px} + \frac{1}{Pw} + \&c.$$

But by the Algebra, Art. 525. the roots of the equation $x^n - \frac{q}{p}x^{n-1} + \&c. = 0$ are *AB*, *AC*, *AD*, &c. whatever be the angle at *P*; hence (Algebra, Art. 267.) the coefficients of $x^n - \frac{q}{p}x^{n-1} + \&c.$ are conſtant; and if *P* be aſſumed a fixed point, x is invariable; hence $x^n - \frac{q}{p}x^{n-1} + \&c.$ is conſtant, conſequently $n x^{n-1} - \overline{n-1}.\frac{q}{p}x^{n-2} + \&c.$ is conſtant; therefore the ſum of the reciprocal ſubtangents is a conſtant quantity.

Prop. CI.

Given the arc of a circle; to find it's ſine and coſine.

185. Put the radius $OA = r$, the arc $AB = z$, it's ſine $BE = x$, coſine $OE = y$, and produce *BE* to *D*; then then Art. 46. $\dot{z} : -\dot{y} :: r : x = \frac{-r\dot{y}}{\dot{z}}$. Now correſponding to the ſame value *OE* of y, z may be either *AB* or *AD*; but the arc beginning at *A*, if we conſider *AB* as poſitive, *AD* will be negative, therefore every poſitive value of z * has a negative value equal

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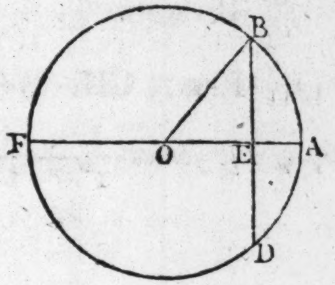
* If every poſitive value of z have a negative value equal to it, the equation whoſe roots are thoſe values of z will have only the even powers of z; for if $z = a$, $z = -a$, then $z - a = 0$, $z + a = 0$, and conſequently the quadratic from theſe two will be $z^2 - a^2 = 0$; and as every ſuch pair of roots will form a ſimilar quadratic, it is manifeſt that the equation formed for the multiplication of theſe quadratics will contain only the even powers of z.

to it; hence, by the note, if we aſſume y in a ſeries of the powers of z, only the even powers of z will enter.



Aſſume therefore $y=r+az^2+bz^4+cz^6+$&c. the firſt term being r, becauſe when $z=0$, $y=r$; hence $\dot{y}=2az\dot{z}+4bz^3\dot{z}+6cz^5\dot{z}+$&c. therefore $x\left(=\frac{-r\dot{y}}{\dot{z}}\right)=-2raz-4rbz^3-6rcz^5-$&c. hence $\dot{x}=-2ra\dot{z}-3.4rbz^2\dot{z}-5.6rcz^4\dot{z}-$&c. But (Art. 46.) $\dot{z}:\dot{x}::r:y$, hence $y\dot{z}=r\dot{x}$, and $y\dot{z}-r\dot{x}=0$; now in this equation, inſtead of y and $\dot{x}$ ſubſtitute their values above found, and we have

$$\left.\begin{array}{l} r\dot{z}+\quad az^2\dot{z}+\quad bz^4\dot{z}+\text{\&c.} \\ 2r^2a\dot{z}+3.4r^2bz^2\dot{z}+5.6r^2cz^4z+\text{\&c.} \end{array}\right\}=0;$$

hence (Art. 110.) $2r^2a+r=0$, $3.4r^2b+a=0$, $5.6r^2c+b=0$, &c. conſequently $a=\frac{-1}{2r}$; $b=\frac{-a}{3.4r^2}=\frac{1}{2.3.4r^3}$; $c=\frac{-b}{5.6r^2}=\frac{-1}{2.3.4.5.6r^5}$; &c. hence $y=r-\frac{z^2}{2r}+\frac{z^4}{2.3.4r^3}-\frac{z^6}{2.3.4.5.6r^5}+$&c. Alſo $-2ra=1$;

$=1$; $-4rb=\frac{-1}{2.3r^2}$; $-6rc=\frac{1}{2.3.4.5r^4}$; &c. hence

$$x=z-\frac{z^3}{2.3r^2}+\frac{z^5}{2.3.4.5r^4}-\&c.$$

PROP. CII.

To find the ſum of the ſeries $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. *ad infinitum.*

186. Put the radius $AO=1$, $EB=x$, $AB=z$; then (Art. 185.) $x=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. Let $x=0$, and then $z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-\&c.=0$, or $1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-\&c.=0$, the former equation containing one root $=0$, it being diviſible by z, or $z=0$, (Elem. Alg. Art. 266.) which is taken away by dividing by z. But if $c=$ the ſemi-circumference of the circle, the other values z, correſponding to $x=0$, will be $1c$, $2c$, $3c$, &c. ad infinitum, and by taking the arcs in a contrary direction, $-1c$, $-2c$, $-3c$, &c. ad infinitum (Elem. Alg. 473.); hence theſe values of z are the roots of the equation $1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-\&c.=0$. Put $z=\frac{1}{y}$, and the equation becomes $1-\frac{1}{2.3.y^2}+\frac{1}{2.3.4.5.y^4}-\&c.=0$; multiply it by y^n, and it becomes $y^n-\frac{y^{n-2}}{2.3}+\frac{y^{n-4}}{2.3.4.5}-\&c$

$=0$,

$=0$, which equation contains n roots $=0$, the other roots remaining the ſame. But as $y=\frac{1}{z}$, the values of y are $\frac{1}{1c}, \frac{1}{2c}, \frac{1}{3c}$, &c. and $-\frac{1}{1c}, -\frac{1}{2c}, -\frac{1}{3c}$, &c. ad inf. Now (Alg. Art. 349.) the ſum of the ſquares of the roots of the laſt equation is $\frac{1}{3}$; and the ſquares of the poſitive values of y being the ſame as the ſquares of the negative values, we have $\frac{2}{1^2c^2}+\frac{2}{2^2c^2}+\frac{2}{3^2c^2}+$ ad inf. $=\frac{1}{3}$, conſequently $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. ad inf. $=\frac{c^2}{6}$.

Cor. 1. In like manner we may find the ſum of any of the *even* powers of the reciprocals of the natural numbers, by aſſuming the ſum equal to it's value given by the ſame Art. in the Algebra. For inſtance, the ſum of the fourth powers of the roots of the equation is $\frac{1}{45}$; hence $\frac{2}{1^4c^4}+\frac{2}{2^4c^4}+\frac{2}{3^4c^4}+$&c. $=\frac{1}{45}$, conſequently $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+$ &c. $=\frac{c^4}{90}$.

The ſum of the reciprocals of the *odd* powers cannot be found by this method, becauſe the ſum of the *odd* powers of the *negative* roots deſtroy thoſe of the *poſitive*.

Cor. 2. By tranſpoſition, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+$ &c. $=\frac{c^2}{6}-\frac{1}{2^2}-\frac{1}{4^2}-$ &c. $=\frac{c^2}{6}-\frac{1}{2^2}\times\overline{\frac{1}{1^2}+\frac{1}{2^2}+\text{\&c.}}=\frac{c^2}{6}-$

$\frac{1}{2^2} \times \frac{c^2}{6} = \frac{c^2}{8}$. And in like manner we may find the ſum of the reciprocals of all the even powers of 1, 3, 5, &c.

Prop. CIII.

Suppoſing the force of gravity to vary as the n^{th} power of the diſtance from the center of the earth, and the compreſſive force of the air to vary as it's denſity; to find the denſity of the air at any altitude above the ſurface of the earth.

187. Let the radius of the earth $=1$, $x =$ the diſtance of any point above the earth's ſurface from the center, $v =$ the denſity of the air at that point, the denſity at the ſurface being unity; $h =$ the altitude of an homogeneous atmoſphere. Now it appears by experiment, that the compreſſive force of the air varies as it's denſity; conſequently the fluxion of the compreſſive force muſt be to the fluxion of the denſity as the compreſſive force is to the denſity, and this ratio is the ſame at all altitudes. Now at any diſtance x from the earth's center, the fluxion of the compreſſive force muſt be in proportion to the force of gravity, the denſity and the fluxion of the altitude; hence $x^n v \dot{x}$ is in a conſtant ratio to $-\dot{v}$, writing the latter fluxion with the ſign $-$ (Art. 16.) becauſe v decreaſes as x increaſes; according to this repreſentation of the compreſſive force, h will repreſent the compreſſive force at the ſurface; hence $h : 1 :: x^n v \dot{x} : -\dot{v}$, therefore $x^n \dot{x} = -h \times \frac{\dot{v}}{v}$, and $\frac{x^{n+1}}{n+1} = -h \times \text{h. l.} v + C$;

$+C$; but when $x=1$, $v=1$, and this equation becomes $\frac{1}{n+1}=C$; hence the corrected fluent $\frac{x^{n+1}}{n+1} = -h \times \text{h. l. } v + \frac{1}{n+1}$, conſequently $\frac{1-x^{n+1}}{n+1} = h \times \text{h. l. } v$, an equation expreſſing the relation between the altitude and denſity.

Cor. 1. If we ſuppoſe the force to vary inverſely as the ſquare of the diſtance, n becomes -2; hence $\frac{1}{x} - 1 = h \times \text{h. l. } v$; if therefore x increaſe in muſical progreſſion, $\frac{1}{x}$ will decreaſe in arithmetic progreſſion, and conſequently the h. l. v will decreaſe in arithmetic progreſſion.

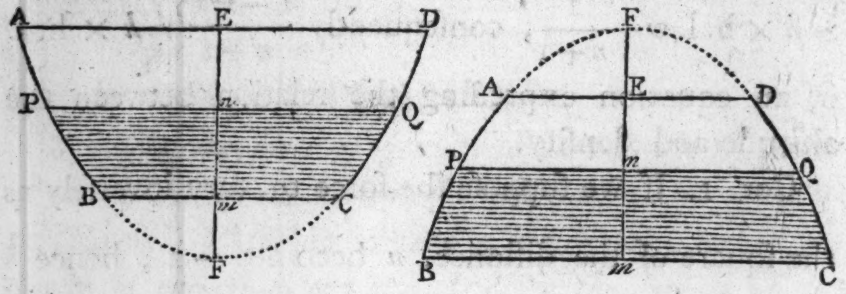
Cor 2. If the force of gravity be ſuppoſed conſtant, $n=0$; hence $1-x=h \times \text{h. l. } v$; and if x increaſe in arithmetic progreſſion, then $1-x$ will decreaſe in arithmetic progreſſion, conſequently the h. l. v will decreaſe in arithmetic progreſſion.

PROP. CIV.

To find the time in which a veſſel ABCD filled with fluid will be in emptying itſelf through a very ſmall orifice m at the bottom.

188. Put $a = 32\frac{1}{6}$ feet $= 386$ inches, $x = mn$ the depth of the fluid at any point of time, $z =$ the area of the ſurface PQ of the fluid, $m =$ the area of the orifice, $t =$ the time in which the ſurface deſcends from PQ to BC. Now it appears by experiment, that the velocity of the fluid at the ſection of the ori-

fice is that which a body acquires in falling down $\frac{1}{2}x$, ſuppoſing the orifice to be very ſmall compared with



the ſurface of the fluid; hence, by mechanics, $\sqrt{\frac{1}{2}a} : \sqrt{\frac{1}{2}x} :: a : \sqrt{ax}$ = the velocity (per ſecond) at the orifice; and, by the Principles of Hydroſtatics, $z : m :: \sqrt{ax} : \frac{m}{z} \times \sqrt{ax}$ the velocity with which the ſurface deſcends; hence (Art. 72.) $\dot{t} = \frac{\dot{x}}{\frac{m}{z} \times \sqrt{ax}} = \frac{z\dot{x}}{m\sqrt{ax}}$, the fluent of which (corrected when neceſſary) gives t.

Examples.

Ex. 1. Let the veſſel be a cylinder or priſm, and put $h = Em$ it's altitude. In this caſe z is conſtant, and $\dot{t} = \frac{z\dot{x}}{m\sqrt{ax}} = \frac{z}{m\sqrt{a}} \times x^{-\frac{1}{2}}\dot{x}$, whoſe fluent is $t = \frac{2zx^{\frac{1}{2}}}{m\sqrt{a}} = \frac{2z}{m} \times \sqrt{\frac{x}{a}}$, which wants no correction; and when $x = h$, $t = \frac{2z}{m} \times \sqrt{\frac{h}{a}}$ the time of the emptying.

Ex. 2. Let $ABCD$ be the fruſtrum of a cone, and put $FE=c$, $EA=d$, $Em=e$, $p=3{,}14159$ &c. then $Fn=c\pm x$, the ſign + or − being taken according as the greater or leſſer end is downwards; and (FA, FD being now right lines) by ſimilar triangles, $c:d::c\pm x:Pn=\frac{d}{c}\times\overline{c\pm x}$; hence $z=\frac{pd^2}{c^2}\times\overline{c\pm x}^2$; conſequently $\dot{t}=\frac{pd^2}{mc^2\sqrt{a}}\times x^{-\frac{1}{2}}\times\overline{c\pm x}^2\times\dot{x}=\frac{pd^2}{mc^2\sqrt{a}}\times\overline{c^2x^{-\frac{1}{2}}\dot{x}\pm 2cx^{\frac{1}{2}}\dot{x}+x^{\frac{3}{2}}\dot{x}}$, and $t=\frac{pd^2}{mc^2\sqrt{a}}\times\overline{2c^2x^{\frac{1}{2}}\pm\frac{4}{3}cx^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}}$, which requires no correction; and when $x=e$, $t=\frac{pd^2}{mc^2\sqrt{a}}\times\overline{2c^2e^{\frac{1}{2}}\pm\frac{4}{3}ce^{\frac{3}{2}}+\frac{2}{5}e^{\frac{5}{2}}}$, the whole time of emptying.

If the orifice be a circle whoſe radius $=r$, then $m=pr^2$; conſequently $t=\frac{d^2}{r^2c^2\sqrt{a}}\times\overline{2c^2e^{\frac{1}{2}}\pm\frac{4}{3}ce^{\frac{3}{2}}+\frac{2}{5}e^{\frac{5}{2}}}$.

Cor. If the baſe be downwards, and we take the whole cone, we muſt put $c:d::Fm:mB$, $\therefore\ \frac{d^2}{c^2}=\frac{mB^2}{mF^2}$; and as $c=0$, we have $t=\frac{mB^2}{r^2\times mF^2\times\sqrt{a}}\times\frac{2}{5}mF^{\frac{5}{2}}=\frac{2mB^2\times\sqrt{mF}}{5r^2\times\sqrt{a}}$, the whole time of emptying.

If the vertex be downwards, and the orifice be ſo ſmall that we may conſider Em as equal to EF, or $e=c$; then $t=\frac{d^2}{rc^2\sqrt{a}}\times\frac{16}{15}c^{\frac{5}{2}}=\frac{16d^2\sqrt{c}}{15r^2\sqrt{a}}$, the whole time of emptying.

Ex. 3. Let *BFC* be an hemiſphere ſtanding on it's bottom; put the radius $mB = mF = r$; then $Pn^2 = r^2 - x^2$, and $z = p \times \overline{r^2 - x^2}$; hence $\dot{t} = \frac{p \times \overline{r^2 - x^2} \times \dot{x}}{m\sqrt{ax}} = \frac{p}{m\sqrt{a}} \times \overline{r^2 x^{-\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x}}$, whoſe fluent is $t = \frac{p}{m\sqrt{a}} \times \overline{2r^2 x^{\frac{1}{2}} - \frac{2}{5} x^{\frac{5}{2}}}$, which wants no correction; and when $x = r$, $t = \frac{8p}{5m\sqrt{a}} \times r^{\frac{5}{2}}$, the whole time of emptying.

If the orifice be a circle whoſe radius is *w*, then $m = pw^2$; hence $t = \frac{8r^{\frac{5}{2}}}{5w^2\sqrt{a}}$.

If the hemiſphere ſtand on it's vertex, $Pn^2 = 2rx - x^2$; hence $z = p \times \overline{2rx - x^2}$, conſequently $\dot{t} = \frac{p}{m\sqrt{a}} \times \overline{2ax^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x}}$, whoſe fluent is $t = \frac{p}{m\sqrt{a}} \times \overline{\frac{4}{3} rx^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}}}$, which requires no correction; and when $x = r$, $t = \frac{14pr^{\frac{5}{2}}}{15m\sqrt{a}} = \frac{14r^{\frac{5}{2}}}{15w^2\sqrt{a}}$, the whole time of emptying.

Ex. 4. Let *BFC* be a paraboloid ſtanding on it's bottom; put it's parameter $= r$, altitude $FM = e$, then $r \times \overline{e - x} = Pn^2$, and $pr \times \overline{e - x} = z$; hence $\dot{t} = \frac{pr}{m\sqrt{a}} \times \overline{ex^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x}}$, whoſe fluent is $t = \frac{pr}{m\sqrt{a}} \times \overline{2ex^{\frac{1}{2}} - \frac{2}{3} x^{\frac{3}{2}}}$, which requires no correction; and when $x = e$, $t = \frac{4pre^{\frac{3}{2}}}{2m\sqrt{a}} = \frac{4re^{\frac{3}{2}}}{3w^2\sqrt{a}}$ the whole time of emptying.

If the paraboloid ſtand on its vertex, $Pn^2 = rx$; hence $z = prx$; conſequently $\dot{t} = \frac{prx^{\frac{1}{2}}\dot{x}}{m\sqrt{a}}$, and $t = \frac{2prx^{\frac{3}{2}}}{3m\sqrt{a}}$, which wants no correction; and when $x = e$, $t = \frac{2pre^{\frac{3}{2}}}{3m\sqrt{a}} = \frac{2re^{\frac{3}{2}}}{3w^2\sqrt{a}}$ the whole time of emptying.

In like manner, whatever be the form of the veſſel, we may find the time of emptying; ſubſtituting into the value of $\dot{t}$, the quantity z expreſſed in terms of x, and then taking the fluent.

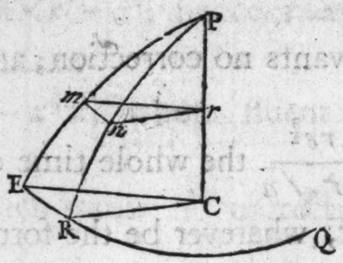
ON MERCATOR's PROJECTION.

Prop. CV.

If P be the pole of the earth, EQ the equator, PE, PR, two meridians, mn a ſmall circle parallel to ER; then the length of a degree of latitude : the length of a degree of longitude at m :: radius : the coſine of the latitude of m, ſuppoſing the earth to be a ſphere.

189. For let PC be the radius of the earth; draw mr, nr perpendicular to it, and join EC, RC. Then mr, nr being parallel to EC, RC reſpectively, the angle $mrn = ECR$; hence by ſimilar ſectors, $ER : mn ::$ EC

EC : *mr* coſine of *m E*. But when the angle is given, the length of an arc of a degree is in proportion to the radius; alſo the length of a degree of the great circle *ER* is a degree of latitude, and the length of a degree of *mn* is a degree of longitude at *m*; hence a degree of latitude : a degree of longitude :: radius : the coſine of latitude.

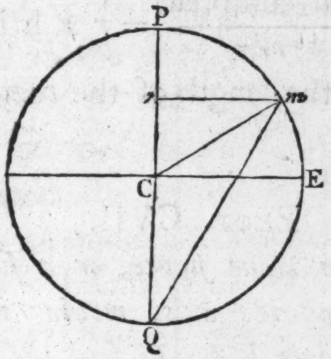


In MERCATOR's Projection, the ſphere is projected upon a plane, and the meridians *EP*, *RP* are ſtraight lines parallel to each other; conſequently *P* muſt be at an infinite diſtance from the equator *EQ*. In this caſe, the arc *mn* being the ſame at all latitudes, the length of a degree of longitude is every where the ſame; to preſerve therefore the proper proportion between the degrees of latitude and longitude, the degrees of latitude muſt increaſe as you go from the equator, ſo that they may always be to the degrees of longitude, in the proportion of radius to the coſine of latitude.

PROP. CVI.

In this projection, it is required to find the length of an arc of the meridian, correſponding to any given latitude.

190. Let P be the pole, E the equator, PCQ a diameter of the earth, C the center; m any place on the ſurface; draw mr perpendicular to PQ, and join mC, mQ. Put $Cm=r$, $Em=x$, Cr (the ſine of Em the



latitude of m) $=y$, and the length of Em on the projection $=z$, called the *meridional parts*. Then by Prop. 105. $\sqrt{r^2-y^2}$ (coſ. of lat.) : $r :: \dot{x} : \dot{z} = \frac{r\dot{x}}{\sqrt{r^2-y^2}}$ but (Art. 46.) $\dot{x} = \frac{r\dot{y}}{\sqrt{r^2-y^2}}$; hence $\dot{z} = \frac{r^2\dot{y}}{r^2-y^2} = \frac{r}{2} \times \frac{2r\dot{y}}{r^2-y^2}$, ∴ $z = \frac{r}{2} \times$ h. l. $\frac{r+y}{r-y} + C$ (Art. 45. Ex. 6.) $= r \times$ h. l. $\sqrt{\frac{r+y}{r-y}} + C$, by the nature of logarithms. But by plane trig. $\sqrt{r^2-y^2}$ (mr) : $r+y$ (rQ) :: r (rad.) : $\frac{r \times \overline{r+y}}{\sqrt{r^2-y^2}}$ $= r\sqrt{\frac{r+y}{r-y}}$ the tangent of the angle rmQ = cotangent of rQm = cotan. of $\frac{1}{2}\,rCm$ = cotan. $\frac{1}{2}$ the complement of lat.; hence $\sqrt{\frac{r+y}{r-y}} = \frac{\text{cotan.}\,\frac{1}{2}\,\text{comp. lat.}}{r}$;

conſequently $z = r \times \text{h. l.} \frac{\text{tan. } \frac{1}{2} \text{ comp. lat.}}{r} + C$; but when $z = 0$, cotan. ½ comp. lat. $= r$; hence $0 = r \times \text{h. l.} \frac{r}{r} + C = r \times \text{h. l. } 1 + C = 0 + C$, $\therefore C = 0$; conſequently

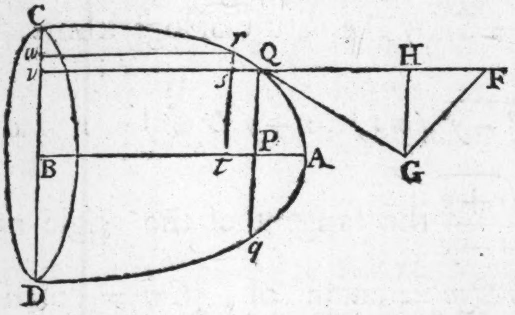
$$z = r \times \text{h. l.} \frac{\text{cotan.} \frac{1}{2} \text{comp. lat.}}{r} = r \times \text{h. l. cotan.} \tfrac{1}{2} \text{ comp. lat.} - r \times \text{h. l. } r,$$

the length of the meridian *Em* in the projection.

PROP. CVII.

Let CAD be a plane figure, or a ſolid generated by it's revolution about it's axis, moving in a fluid in the direction of it's axis BA; to find the reſiſtance of the curve line CAD, or of the ſurface of the ſolid, to the reſiſtance on the baſe CD.

191. Draw *FQsv* and *wr* parallel to *AB*, *rst*, *QPq* perpendicular to *AB*, then if $AP = x$, $PQ = y$, $PA = z$, it appears from Art$^{\text{s}}$. 23. and 27. that ultimately, by bringing *r* up to *Q*, $Qs = \dot{x}$, $sr = \dot{y}$, $Qr = \dot{z}$. Draw the tangent *QG* and let fall the perpendicular *FG* upon it, and alſo *GH* upon *FQ*. Now let *FQ* repreſent the force of one particle of the fluid, then



then if that particle ſtruck the baſe at v it's whole force would act to oppoſe the motion, becauſe it acts perpendicular to the baſe, and therefore no part of it's force is loſt; but ſtriking the curve at Q obliquely, if the force FQ be reſolved into GQ and FG, then GQ is here ſuppoſed to be loſt by the obliquity of the ſtroke, and FG is the only effective part; but this not being oppoſite to the motion of the body, we muſt reſolve it into FH and HG, and then FH is that part which oppoſes the motion of the body, and HG is deſtroyed by an equal and oppoſite force of a particle acting at q. Hence the force of a particle at v : force at Q :: $FQ : FH$ :: (becauſe $FQ : FG :: FG : FH$) $FQ^2 : FG^2$ :: (by ſim. trian.) $\dot{z}^2 : \dot{y}^2$. Now the quantity of fluid ſtriking Qr and vw is the ſame, and in proportion to sr or $\dot{y}$. Hence if we conſider it as a *plane* figure, the force againſt vw : force againſt Qr :: $\dot{y}\dot{z}^2 : \dot{y}^3$ (being as the number of particles × force of each) :: $\dot{y} : \frac{\dot{y}^3}{\dot{z}^2} = \frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2} = \frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$; hence the whole reſiſtance on the baſe : that on the curve :: the fluent of $\dot{y}$, or y, : fluent (F) of $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$. For a *ſolid*, the number of particles ſtriking the area generated by vw will be as $vw \times$ circum. deſcribed by v, or as $vw \times y$ or as $y\dot{y}$; hence, for the ſame reaſon, the reſiſtance on the baſe : that on the ſurface :: the flu. of $y\dot{y}$, or $\frac{1}{2}y^2$, : flu. (F) of $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$.

Ex. 1. If AC, CD be ſtraight lines, the plane is a triangle, and $\dot{x} : \dot{y} :: x : y :: a\ (AB) : b\ (BC)$, $\therefore \frac{\dot{x}^2}{\dot{y}^2} = \frac{a^2}{b^2}$; hence the reſiſtances are as y : flu. $\frac{\dot{y}}{1+\frac{a^2}{b^2}} :: y : \frac{y}{1+\frac{b^2}{a^2}} :: b^2+a^2 : b^2 :: AC^2 : BC^2$. The ſame is true for the cone, or for any priſmatic ſolid.

Ex. 2. Let CAD be a ſemicircle; put $AB=r$, then $y^2=2rx-x^2$, hence $\dot{x} = \frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{r^2-y^2}}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{y^2}{r^2-y^2}$; $\therefore \dot{F} = \frac{\dot{y}}{1+\frac{y^2}{r^2-y^2}} = \frac{r^2\dot{y}-y^2\dot{y}}{r^2}$, $\therefore F=y-\frac{y^3}{3r^2}$; hence the reſiſtances are as $y : y-\frac{y^3}{3r^2}$, which, when $y=r$, is as $3 : 2$.

Ex. 3. Let CAD be an hemiſphere, then $\dot{F} = \frac{y\dot{y}}{1+\frac{y^2}{r^2-y^2}} = \frac{r^2y\dot{y}-y^3\dot{y}}{r^2}$, $\therefore F=\frac{1}{2}y^2-\frac{y^4}{4r^2}$; hence the reſiſtances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2-\frac{y^4}{4r^2}$, which, when $y=r$, is as $2 : 1$.

Ex. 4. Let the ſolid CAD be generated by the cycloid AC about AB, BC being the axis of the cycloid; then if $a=BC$, $y=z-\frac{z^2}{4a}$ by the nature of the curve; hence $\dot{y}=\dot{z}-\frac{z\dot{z}}{2a}$, $\therefore \dot{F}=\frac{y\dot{y}\times\dot{y}^2}{\dot{z}^2}=\frac{y\dot{y}\times\overline{2a-z}^2}{4a^2}=$

$y\dot{y}\times$

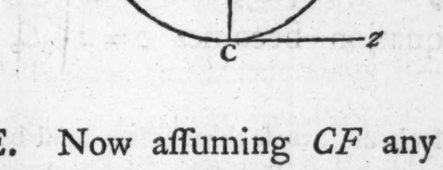
$\frac{y\dot{y}\times\overline{a-y}}{a}=y\dot{y}-\frac{y^2\dot{y}}{a}$, and $F=\frac{1}{2}y^2-\frac{y^3}{3a}$; hence the reſiſtances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2-\frac{y^3}{3a}$, which, when $y=a$, is as $3 : 1$.

Conſidering the body as a ſolid, and the force of a particle on the baſe as conſtant, the force of a particle on the ſurface $\propto \frac{\dot{y}^2}{\dot{z}^2}$, and the area generated by rs being as $y\dot{y}$, the reſiſtance againſt $QR \propto \frac{y\dot{y}^3}{\dot{z}^2}$.

Prop. CVIII.

If a perfectly flexible chain ACB of uniform denſity and magnitude be hung upon two pins at A and B; to find the curve which it will form itſelf into.

192. Let C be the loweſt point, draw the axis CD perpendicular to the horizon; draw alſo EF, Gn perpendicular to CD, Fn a tangent at F, and Fm perpendicular to FE. Now aſſuming CF any part of the chain, we may conſider this part as if it were perfectly rigid; for conceive CF to become perfectly rigid, and it



it is manifeſt that no alteration whatever can take place; for the gravity of the chain gives *CF* a certain ſituation; and if we make that part to become inflexible, we add no new force; we only ſuppoſe a coherence to take place between the conſtituent particles whilſt they are ſo diſpoſed. Conſidering therefore *CF* as a perfectly inflexible body, it is kept at reſt by three forces; at *C* by the action of the part *BC* of the chain in the direction *Cz* of the tangent at *C*; at *F* by the action of the part *FA* of the chain in the direction *Fn* of the tangent at *F*; and by it's gravity in a direction parallel to *EC*;* but *Cz* is parallel to *mn*, and *CE* to *mF*; hence theſe three forces act parallel to the three ſides of the triangle *Fmn*, and conſequently will be reſpectively proportional to them, the body *FC* being at reſt. Put $CE=x$, $EF=y$, $CF=z$, then (Art. 23. and 27.) $mn=\dot{x}$, $Fm=\dot{y}$, $Fn=\dot{z}$. Now the chain being of uniform denſity and magnitude, the gravity of any part *CF* will be in proportion to it's length z; alſo let $a=$ the tenſion of the chain *BC* at *C* acting in the direction *Cz*, a conſtant quantity, it not varying by changing the point *F*. Hence $a : z :: \dot{y} : \dot{x}$, $\therefore a\dot{x}=z\dot{y}$; but $\dot{z}^2=\dot{x}^2+\dot{y}^2=\dot{x}^2+\frac{a^2\dot{x}^2}{z^2}$, therefore $z^2\dot{z}^2=z^2\dot{x}^2+a^2\dot{x}^2$, conſequently $\dot{x}=\frac{z\dot{z}}{\sqrt{a^2+z^2}}$, whoſe fluent (Art. 39.) is $x=\sqrt{a^2+z^2}+C$; but when $x=0$, then $z=0$, hence the equation becomes $0=a+C$, $\therefore C=-a$;

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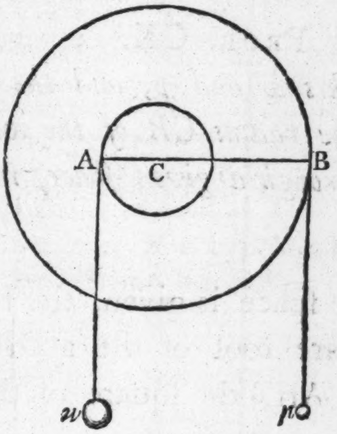
* As, by Mechanics, theſe three forces muſt be directed to one point, if the two tangents *nF*, *zC* be produced to meet, the interſection muſt be in the line of direction paſſing through the center of gravity of *FC*.

therefore the correct fluent is $x=\sqrt{a^2+z^2}-a$, and by tranſpoſing a and ſquaring both ſides, $x^2+2ax=z^2$, the equation of the curve.

Prop. CIX.

Given the radii BC, AC of a wheel and axis, and the weight p which draws up w; to find w, ſo that the momentum communicated to it in a given time may be a maximum, the wheel and axis being ſuppoſed of no weight.

193. Put $BC=b$, $AC=a$; then, by Mechanics, the forces with which w and p endeavour to deſcend are aw and bp; hence the moving force is as $bp-aw$; alſo



the inertia of each weight is (Art. 95.) as $a^2\times w$ and $b^2\times p$; hence the accelerative force of the lever is as $\frac{bp-aw}{b^2p+a^2w}$; and as the acceleration of any point of a lever muſt (beſides the accelerating force with which the lever itſelf is made to revolve) be in proportion to the diſtance of that point from the fulcrum, the accelerative force of the point A, or of w, will be as $\frac{abp-a^2w}{b^2p+a^2w}$, which is as the velocity generated in w in a given time, conſequently the momentum of w will be as $\frac{abp-a^2w}{b^2p+a^2w}\times w=\frac{abpw-a^2w^2}{b^2p+a^2w}$ = a maximum, or $\frac{bpw-aw^2}{b^2p+a^2w}$ = a maximum; hence (Art. 21.) it's fluxion

$$\frac{\overline{bp\dot{w}-2aw\dot{w}}\times\overline{b^2p+a^2w}-a^2\dot{w}\times\overline{bpw-aw^2}}{\overline{b^2p+a^2w}|^2}=0,\text{ or}$$

$\overline{bp-2aw}\times\overline{b^2p+a^2w}-a^2\times\overline{bpw-aw^2}=0$, hence $w=\overline{\sqrt{\frac{b^4}{a^4}+\frac{b^3}{a^3}}-\frac{b^2}{a^2}}\times p$.

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If $a=b$, $w=\overline{\sqrt{2}-1}\times p$.

PROP. CX.

Given two weights w and p, and the radius CA of the axle, to find the radius CB of the wheel, ſo that p may draw up w through a given ſpace, in the leaſt time poſſible.

194. When the ſpace is given, the time varies inverſely as the ſquare root of the accelerative force; hence (by the laſt Art.) the ſquare of the time varies as $\frac{b^2p+a^2w}{abp-a^2w}$ a maximum, where b is variable; put it's fluxion $=0$, and we get $b=\frac{aw}{p}+\frac{\sqrt{a^2w^2+a^2pw}}{p}$.

If $p=w$, $b=a\times\overline{1+\sqrt{2}}$.

THE END OF VOL. II.

CORRIGENDA.

Page 60, line laſt, for $a :: y$ read $a :: \dot{y}$.

P. 153, l. 14, for $\frac{}{4}$ read $\frac{1}{4}$,

Prop. XCIV, for x *being equal to or leſs than unity*, read, x *being equal to* $+1$ *or* -1, *or between them.*

To the enunciation of Prop. XCV, add, x *being the ſame as in the laſt propoſition.*

11, l. 16, *for* $2xy$, *read* $2\dot{x}\dot{y}$.
95, l. 8. *for* RN, *read* LN.
136, l. 4, *for* as w *read* as $\dot{w}$.
160, in Art. 110, Z is hyp. log. z.
161, in Art 115, *for* Z *read* V.
 l. 17, *for* F *read* $\dot{F}$.
 l. 18, *for* $\frac{nb^2}{2}$ *read* $\frac{2}{nb^2}$.
180, l. 12, *for* $\frac{-\overline{m-1}\times b\dot{x}}{n}$ *read* $\frac{-\overline{m-1}\times b\dot{x}}{x}$.
183, l. 14, *for* for *read* from.
185, l. laſt but one, *for* x read $\dot{x}$.
190, l. laſt but one, *for* $\frac{1}{2.3.4}$ *read* $\frac{1}{1.2.3}$.
191, l. 2, *for* $\frac{1}{3.4.5}$ *read* $\frac{1}{2.3.4}$.